Basic notations

- Norms in $L^r \equiv L^r(\mathbb{R}^n)$ are denoted by $\|\cdot\|_r$, $1 \leq r \leq \infty$
- Pairs of conjugate Hölder exponents are denoted by r and $\bar{r}$:

  $1/r + 1/\bar{r} = 1$. Conventionally $2 \leq r \leq \infty$ ..

  $\delta(r) \equiv n/2 - n/r \leq n/2$

- $H^1 \equiv H^1(\mathbb{R}^n) = \{ u \in L^2 : \|u; H^1\|^2 = \|u\|_2^2 + \|\nabla u\|_2^2 < \infty \}$

I interval, X Banach space

$\rightsquigarrow$ $\mathcal{C}(I, X)$ continuous functions from I to X.

$\rightsquigarrow$ $L^q(I, X)$, $1 \leq q \leq \infty$ $\left(L^q_{loc}(I, X) \right)$

$\equiv$ "measurable" functions from I to X such that

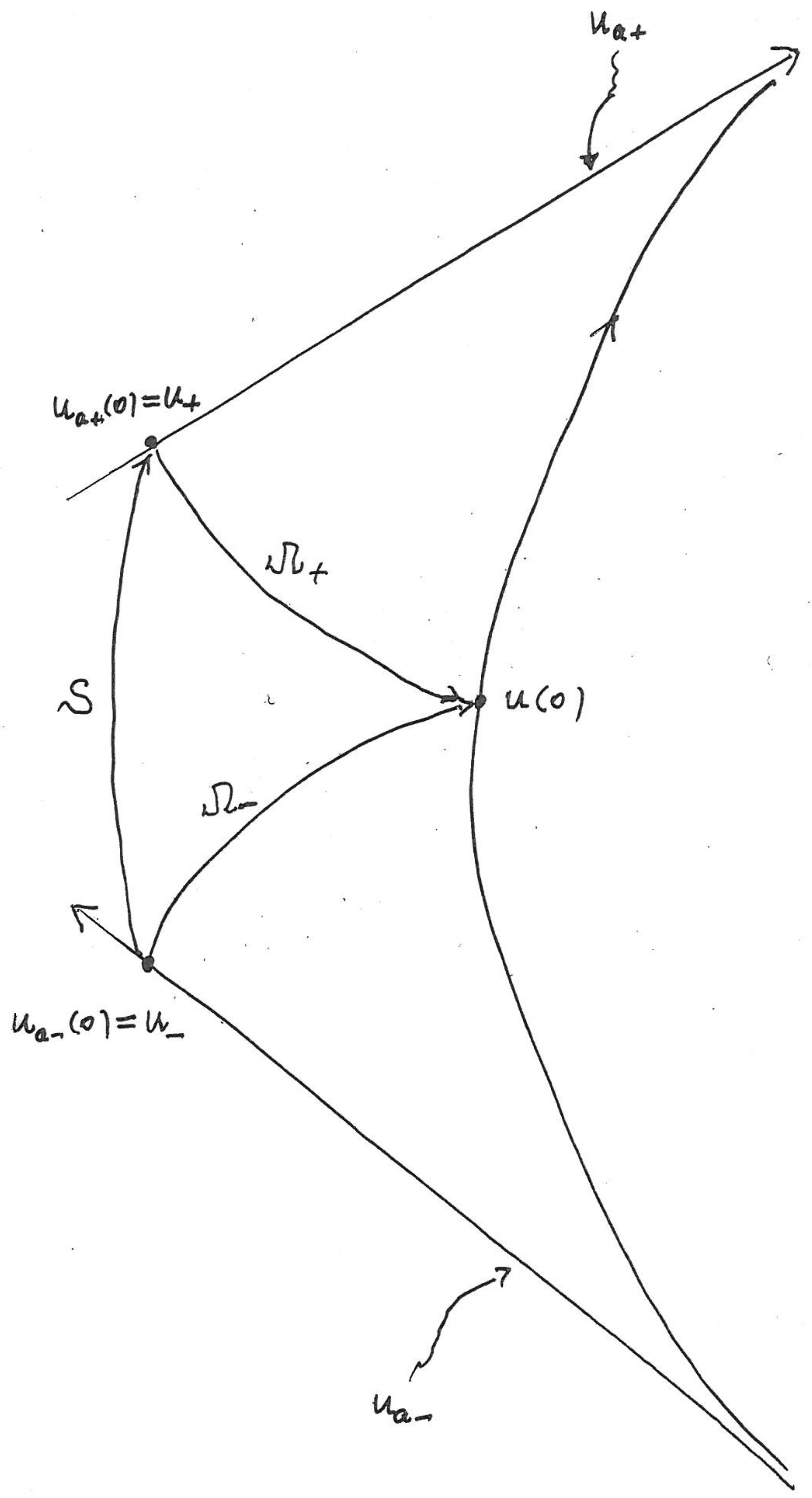
$\|u(\cdot); X\| \in L^q(I)$ $\left(\|u(\cdot); X\| \in L^q_{loc}(I) \right)$

$L^q(I, X)$ is Banach space with $\|u; L^q(I, X)\| = \big\| \|u(\cdot); X\|; L^q(I) \big\|$

If $X = L^r$ with $r < \infty$, and $q < \infty$, then $\mathcal{C}_0^\infty(I \times \mathbb{R}^n)$ is dense in $L^q(I, L^r)$

Basically if $r, q < \infty$ $L^q(I, L^r)$ is the space of functions of space-time $u(t, x)$, $t \in I$, $x \in \mathbb{R}^n$ such that

$$\int_I dt \left(\int dx\, |u(t,x)|^r \right)^{q/r} < \infty .$$



u_{a+}
$u_{a+}(0)=u_+$
Ω_+
S
$u(0)$
Ω_-
$u_{a-}(0)=u_-$
u_{a-}

Equivalence between the differential equation and the integral equation

$$i\partial_t u = -\frac{1}{2}\Delta u + f(u) \equiv i v \qquad \text{(DE)}$$

Def. $U(t) \equiv \exp\left(i \frac{\Delta}{2} t\right)$

$$u(t) = U(t-t_0)u_0 - i\int_{t_0}^{t} dt'\, U(t-t')\, f(u(t')) \qquad \text{(IE)}$$

$$= U(t-t_0)u_0 + (F(u))(t) \equiv (Au)(t)$$

- Preliminary results on the LS equation

Lemma 1. Let $h \in \mathcal{C}(I, H^k)$ for some $k \in \mathbb{R}$ and I open interval.

Suppose $\begin{cases} (i\partial_t + \frac{1}{2}\Delta)h = 0 & (*) \\ h(t_0) = 0 & \text{for some } t_0 \in I \end{cases}$. Then $h(t) = 0\ \forall t$.

Proof. Since $H^k \subset \mathcal{S}'$ we can perform the Fourier transform F in the space variables on the equation $(*)$:

$$\left(i\partial_t - \frac{1}{2}\xi^2\right)(Fh)(t,\xi) = 0$$

$$\Longrightarrow \qquad \partial_t\left(e^{\frac{i\xi^2}{2}t}(Fh)(t,\xi)\right) = 0$$

$\square$

Lemma 2. Let $u_0 \in H^k$ for some $k \in \mathbb{R}$. Then

$$\left(i\partial_t + \frac{1}{2}\Delta\right)U(t)u_0 = 0 \qquad \text{in } \mathcal{D}'(\mathbb{R}^{n+1}).$$

Proof $\quad U(t)u_0 \in \mathcal{C}(\mathbb{R}, H^k) \cap \mathcal{C}^1(\mathbb{R}, H^{k-2})$. Furthermore

$$\left(F\, U(t) u_0\right)(\xi) = e^{-i\frac{\xi^2}{2}t}\, \hat{u}_0(\xi) \qquad \text{so that}$$

$$F\left(i\partial_t + \frac{1}{2}\Delta\right) U(t) u_0 = \left(i\partial_t - \frac{1}{2}\xi^2\right) e^{-i\frac{\xi^2}{2}t}\, \hat{u}_0(\xi) = 0$$

$\square$

- Useful result on integrals of the type of that on the RHS of (IE)

Lemma 3. Let I open interval, let $t_0 \in I$, let $g \in L^1_{loc}(I, H^{k_0})$ for some $k_0 \in \mathbb{R}$. Define

$$G(g, t) \equiv -i \int_{t_0}^{t} dt'\, U(t-t')\, g(t')$$

where the integral is understood in H^{k_0}. Then

1) If $g \in \mathcal{C}(I, H^{k_0})$, $\quad G(g, \cdot) \in \mathcal{C}(I, H^{k_0}) \cap \mathcal{C}^1(I, H^{k_0-2})$.

Furthermore

$$\left(i\partial_t + \frac{1}{2}\Delta\right) G(g, \cdot) = g(\cdot) \qquad (*)$$

in H^{k_0-2} and in $\mathcal{D}'(I \times \mathbb{R}^n)$.

2) For general $g \in L^1_{loc}(I, H^{k_0})$ $(*)$ holds in $\mathcal{D}'(I \times \mathbb{R}^n)$.

Proof 1) The equality $(*)$ holds trivially in H^{k_0-2} for $g \in \mathcal{C}(I, H^{k_0})$.

To prove $(*)$ in $\mathcal{D}'(I \times \mathbb{R}^n)$ consider multiplication of $(*)$ by $\theta \in \mathcal{C}_0^\infty(I)$, test function in the time variable, and then integration in time. By integration by part one obtains

$$-i\int dt\ G(g,t)\,\partial_t\theta(t) = -\frac{1}{2}\int dt\,(\Delta G(g,t))\,\theta(t) + \int dt\ g(t)\,\theta(t)$$

where the integral with the Δ is in H^{k_0-2} while the two other integrals are in H^{k_0}. Dualization with $\varphi \in \mathcal{C}_0^\infty(\mathbb{R}^n)$, test function in the space variables, yields

$$-i\int dt\,\langle G(g,t),\varphi\rangle_{H^{k_0},H^{-k_0}}\,\partial_t\theta(t) = -\frac{1}{2}\int dt\,\langle G(g,t),\Delta\varphi\rangle_{H^{k_0},H^{-k_0}}\,\theta(t) + \int dt\,\langle g(t),\varphi\rangle_{H^{k_0},H^{-k_0}}\,\theta(t).$$

Taking as space-time test functions $\psi = \sum_{\text{finite}}{}_j\ \theta_j \otimes \varphi_j$

one obtains

$$-i\int dt\,\langle G(g,t),\partial_t\psi(t)\rangle_{H^{k_0},H^{-k_0}} = -\frac{1}{2}\int dt\,\langle G(g,t),\Delta\psi(t)\rangle_{H^{k_0},H^{-k_0}} + \int dt\,\langle g(t),\psi(t)\rangle_{H^{k_0},H^{-k_0}}$$

which can be rewritten as

$$\left\langle i\partial_t G(g,\cdot) + \frac{1}{2}\Delta G(g,\cdot) - g(\cdot),\psi\right\rangle_{\mathcal{D}'(I\times\mathbb{R}^n),\,\mathcal{D}(I\times\mathbb{R}^n)} = 0$$

By the density of the family of test functions in $I\times\mathbb{R}^n$ of type ψ we obtain $(*)$ in $\mathcal{D}'(I,\mathbb{R}^n)$.

2) For general $g \in L^1_{loc}(I, H^{k_0})$ let us denote by g_R

$\in \mathcal{C}(I, H^{k_0})$ its regularitation in the time variable:

$$g_R \xrightarrow[R\to]{} g \quad \text{in } L^1_{loc}(I, H^{k_0})$$

From

$$\| G(g_R,t) - G(g,t); H^{k_0} \| \leq \left| \int_{t_0}^{t} dt' \, \| g_R(t') - g(t'); \| \right|$$

it follows that $\forall \ \psi \in \mathcal{C}_0^\infty(I \times \mathbb{R}^n)$

$$\left| \int dt \langle (G(g_R,t) - G(g,t)), \psi(t) \rangle_{H^{k_0}; H^{-k_0}} \right| \leq \left| \int dt \int_{t_0}^{t} dt' \langle g_R(t') - g(t'), \psi(t) \rangle_{H^{k_0}, H^{-k_0}} \right| .$$

That estimate proves that

$$G(g_R, \cdot) \longrightarrow G(g, \cdot) \quad \text{in } \mathcal{D}'(I \times \mathbb{R}^n)$$

and therefore (*) holds in general in $\mathcal{D}'(I \times \mathbb{R}^n)$.

$\square$

- <u>Assumption (A)</u>.. I open interval

- $X_j(I)$, $j = 1,2,..N$, Banach spaces of functions from I to the Banach spaces X_j such that $\bigcap_{j=1}^{N} X_j(I) \subset L^\infty_{loc}(I, L^2)$

- $u \in \bigcap_{j=1}^{N} X_j(I)$

- $\forall u \in \bigcap_{j=1}^{N} X_j(I) \qquad f(u) \in L^1_{loc}(I, H^{k_0})$ for some $k_0 \in \mathbb{R}$.

Proposition 1. Let I and u satisfy (A). Let u be solution of (DE) in $\mathcal{D}'(I \times \mathbb{R}^n)$. Then $u \in \mathcal{C}(I, H^{k_1}) \cap \mathcal{C}_w(I, L^2)$ with $k_1 = \mathrm{Min}(-2, k_0)$. Let $t_0 \in I$ and $u_0 \in L^2$ with $u(t_0) = u_0$.

Then u is solution of (IE) (the integral in (IE) is meant to be a Bochner integral in H^{k_0}). In particular

$$u \in \mathcal{C}(I, L^2) + \mathcal{C}(I, H^{k_0}) \text{ and } \int_{t_0}^{t} dt' \, U(t-t') f(u(t')) \in \mathcal{C}_w(I, L^2)$$

Proof. Rewrite (DE) in the form

$$\partial_t u = v \qquad (*) \qquad \text{in } \mathcal{D}'(I \times \mathbb{R}^n).$$

where $$v = i\left(\tfrac{1}{2}\Delta u - f(u)\right) \in L^\infty_{loc}(I, H^{-2}) + L^1_{loc}(I, H^{k_0})$$

$$\subset L^1_{loc}(I, H^{k_1}) \qquad \text{with} \qquad k_1 = \mathrm{Min}(-2, k_0).$$

Choose test functions $\theta \in \mathcal{C}_0^\infty(I)$ and $\varphi \in \mathcal{C}_0^\infty(\mathbb{R}^n)$. Then from $(*)$

$$\int dt \left\{ \langle u(t), \varphi \rangle_{H^{k_1}, H^{-k_1}} (\partial_t \theta)(t) + \langle v(t), \varphi \rangle_{H^{k_1}, H^{-k_1}} \theta(t) \right\} = 0$$

$$\therefore \quad (H^{k_1}) \int dt \left\{ u(t) \, \partial_t \theta(t) + v(t) \, \theta(t) \right\} = 0 \qquad (**)$$

Define $$w(t) \equiv (H^{(k_1)}) \int_{t_0}^{t} dt' \, v(t') \in \mathcal{C}(I, H^{k_1})$$

By Fubini's theorem (on Bochner integrals) on obtains

$$(H^{k_1})\int dt\left\{ w(t)\partial_t\theta(t) + v(t)\theta(t)\right\} = 0 \qquad (***)$$

Comparison of (**) with (***) yields

$$(H^{k_1})\int dt\left\{ u(t) - w(t)\right\}\partial_t\theta(t) = 0$$

$\therefore$ $u(t) = w(t) + \text{const}$, a.e. in t, with const $\in H^{k_1}$

$\therefore$ $u(\cdot)$ can be chosen to belong to $\mathcal{C}(I, H^{k_1})$.

From that continuity and from $u \in L^\infty_{loc}(I, L^2)$ it follows by general arguments that $u \in \mathcal{C}_w(I, L^2)$ so that $u(t) \in L^2$ $\forall t \in I$

Consider now the (IE) with t_0 and u_0 as in the statement of Proposition 1. By Lemmas 2 and 3 one obtains

$$\left(i\partial_t + \tfrac{1}{2}\Delta\right)A(u) = f(u) \qquad \text{in } \mathcal{D}'(I\times\mathbb{R}^n)$$

with $A(u)$ given by the line below the (IE), so that

$$\begin{cases} \left(i\partial_t + \tfrac{1}{2}\Delta\right)(u - A(u)) = 0 & \text{in } \mathcal{D}'(I\times\mathbb{R}^n) \\ u(t_0) - (Au)(t_0) = 0 \end{cases}$$

By Lemma 1 $u = Au$, i.e. u solves the (IE). The announced continuity follows directly from the form of (IE).

$\square$

Proposition 2. Let I and u satisfy (A). Let $t_0 \in I$ and $u_0 \in L^2$. Let u be solution of (IE) (the integral in (IE) is meant to be in H^{k_0}) which obviously belongs to $\mathcal{C}(I, L^2) + \mathcal{C}(I, H^{k_0})$ and satisfies $u(t_0) = u_0$. Then u is solution of (DE) in $D'(I \times \mathbb{R}^n)$.

Proof. In the same way as in Proposition 1, Lemmas 2 and 3 yield

$$\left(i\partial_t + \frac{1}{2}\Delta\right) A(u) = f(u) \quad \text{in } D'(I \times \mathbb{R}^n)$$

From $u = A(u)$ it follows that u solves the (DE) in $D'(I \times \mathbb{R}^n)$.

$\square$

Proposition 3. Let $I = (T, \infty)$ and u satisfy (A). Let u be solution of (DE) in $D'(I \times \mathbb{R}^n)$ such that $\exists$

$$L^2\text{-}\lim_{t_1 \to \infty} U(-t_1)\, u(t_1) \equiv u_+ \qquad (*)$$

(See Proposition 1 for the regularity of $u(\cdot)$ and the fact that u satisfies (IE))

Then, $\forall\, t \in I$, there $\exists$

$$L^2\text{-}\lim_{t_1 \to \infty} \int_{t_1}^{t} dt'\, U(-t')\, f(u(t')) \equiv \int_{\infty}^{t} dt'\, U(-t')\, f(u(t')) \qquad (**)$$

and u is solution of

$$u(t) = U(t)\, u_+ - i \int_{\infty}^{t} dt'\, U(t-t')\, f(u(t')) \qquad (\mathrm{IE}_\infty)$$

where $\int_{\infty}^{t} dt'\, U(t-t')\, f(u(t')) \in \mathcal{C}_w(I, L^2)$

Proof. u satisfies (IE) which can be rewritten as

$$U(-t)u(t) = U(t_0)u(t_0) - i\int_{t_0}^{t} dt'\, U(-t')\, f(u(t')) \qquad (***)$$

which, by (*), implies (**) $\forall\, t \in I$ and therefore (IE$_\infty$).

The continuity properties of the integral at the RHS of (IE$_\infty$) follow from $u \in \mathcal{C}_w(I, L^2)$ $\square$

Proposition 4. Let $I = [T, \infty)$ and u satisfy (A). Let $\bar{t} \in I$ and let us suppose (**) be satisfied for $t = \bar{t}$. Let $u_+ \in L^2$ and let u be solution of (IE$_\infty$). Then u is solution of (DE) in $\mathcal{D}'(I \times \mathbb{R}^n)$ and satisfies (*).

Proof. From (**) and the identity

$$\int_{t_0}^{t_1} dt'\, U(-t') f(u(t')) + \int_{t_1}^{t} dt'\, U(-t') f(u(t')) = \int_{t_0}^{t} dt'\, U(-t') f(u(t'))$$

we obtain (**) for all $t \in I$ and

$$\int_{\infty}^{t} dt'\, U(-t')\, f(u(t')) - \int_{\infty}^{t_0} dt'\, U(-t')\, f(u(t')) = \int_{t_0}^{t} dt'\, U(-t')\, f(u(t'))$$

which toghether with (IE$_\infty$) for t and t_0 yields (***). By Proposition 2 u satisfies (DE) in $\mathcal{D}'(I \times \mathbb{R}^n)$. Furthermore by taking the $L^2 - \lim_{t_0 \to \infty}$ of (***) we obtain (*).

$\square$

Def. A pair (q,r) is admissible if $\frac{2}{q} = \frac{n}{2} - \frac{n}{r} \equiv \delta(r)$ and

$$0 \leq \delta(r) \begin{cases} \leq 1/2 & \text{for } n=1 & \Longleftrightarrow & 2 \leq r \leq \infty \\ < 1 & \text{for } n=2 & \Longleftrightarrow & 2 \leq r < \infty \\ \leq 1 & \text{for } n \geq 3 & \Longleftrightarrow & 2 \leq r \leq 2n/(n-2) \end{cases}$$

Strichartz estimates

(1) Let $u_0 \in L^2$. Then, for any admissible pair (q,r)

$$U(\cdot)u_0 \in \mathcal{C}(\mathbb{R}, L^2) \cap L^q(\mathbb{R}, L^r) \quad \text{and}$$

$$\| U(\cdot)u_0 ; L^q(\mathbb{R}, L^r) \| \leq c_r \| u_0 \|_2$$

(2) Let I be an interval, let $t_0 \in I$ and let $f \in L^{\bar{q}_1}(I, L^{\bar{r}_1})$

for some admissible pai (q_1, r_1). Then, for any admissible pair (q,r)

$$\int_{t_0}^{t} dt' \, U(t-t') f(t') \in \mathcal{C}(I, L^2) \cap L^q(\mathbb{R}, L^r) \quad \text{and}$$

$$\left\| \int_{t_0}^{t} dt' \, U(t-t') f(t') ; L^q(I, L^r) \right\| \leq c_{r_1} c_r \| f ; L^{\bar{q}_1}(I, L^{\bar{r}_1}) \|$$

Remark. The condition of admissibility can be understood

by a scaling argument.

Scaling argument

- If v is a function of space-time, $\forall \lambda \neq 0$, $\lambda \in \mathbb{R}$ define

$$v^{(\lambda)}(t,x) \equiv v\left(\frac{t}{\lambda^2}, \frac{x}{\lambda}\right)$$

$\therefore$
$$\left(\left(i\partial_t + \frac{1}{2}\Delta\right) v^{(\lambda)}\right)(t,x) = \lambda^{-2}\left(\left(i\partial_t + \frac{1}{2}\Delta\right) v\right)^{(\lambda)}(t,x)$$

$\therefore$ If v solves $\left(i\partial_t + \frac{1}{2}\Delta\right) v = 0$, so does $v^{(\lambda)}$.

- For any $v \in \mathcal{C}(I, L^2)$ solution of $\left(i\partial_t + \frac{1}{2}\Delta\right) v = 0$

the 1) of Strichartz implies

$$\| v ; L^q(I, L^r) \| \leq c \, \| v(0) \|_2$$

and therefore, $\forall \lambda \in \mathbb{R} \setminus \{0\}$

$$\lambda^{n/r + 2/q} \| v ; L^q(\mathbb{R}, L^r) \| = \| v^{(\lambda)} ; L^q(I, L^r) \| \leq c \, \| v^{(\lambda)} \|_2 = c \, \lambda^{n/2} \| v(0) \|_2$$

$$\Longrightarrow \quad \frac{n}{r} + \frac{2}{q} = \frac{n}{2} \qquad \text{admissibility.}$$