Proposition IVP$_\infty$ in L^2 for NLS. $p = 1 + 4/n$, $n \geq 1$. Then

(1) For any $u_+ \in L^2$, there exists $T = T(u_+)$ such that the (IE∞)

$$u(t) = U(t)u_+ - i\int_\infty^t dt'\, U(t-t')\, f(u(t'))$$

with $\quad f(u) = \lambda |u|^{p-1} u \quad , \quad \lambda \in \mathbb{C}$

has a unique solution in $X_{p+1}(I)$ where $I = [T, \infty)$.

(Remark: for $p = 1 + 4/n$, $(p+1, p+1)$ is admissible). Furthermore, for all (q,r) admissible, $u \in L^q(I, L^r)$ and is a continuous function of $u_+ \in L^2$ with values in $L^q(I, L^r)$.

(2) The solution u admits u_+ as asymptotic state in L^2, i.e.

$$\| U(-t)u(t) - u_+ \|_2 = \| u(t) - U(t)u_+ \|_2 \longrightarrow 0 \qquad \text{when } t \to \infty.$$

Proof (1) Def:

$$F(u)(t) \equiv -i \int_\infty^t dt'\, U(t-t')\, f(u(t'))$$

$$A(u)(t) \equiv U(t)u_+ + F(u)(t)$$

We want to solve the equation

$$u = Au$$

Remark: from the following estimates we will see that the integral in F is well defined. It will be seen that, for $T \leq s, t$,

$$\left\| \int_1^t dt'\, U(t-t')\, f(u(t'))\,;\, L^\infty((T,\infty);L^2) \right\| \longrightarrow 0 \qquad \text{when } T \to \infty.$$

We set a contraction scheme, so to get a fixe point in an appropriate Banach space. The basic pointwise estimate for f is:

$$|f(u_1) - f(u_2)| \leq C \left(\operatorname{Max}_{j=1,2} |u_j|^{p-1} \right) |u_1 - u_2|$$

Let (q,r), (q_1,r_1), (q_2,r_2) be admissible pairs. Let $I = [T,\infty)$. By Strichartz

$$\| F(u_1) - F(u_2); L^{q_2}(I, L^{r_2}) \| \leq c \, \| f(u_1) - f(u_2); L^{\bar{q}_1}(I, L^{\bar{r}_1}) \| \leq$$

$$\leq \bar{c} \, \| u_1 - u_2; L^q(I, L^r) \| \operatorname{Max}_{j=1,2} \| u_j ; L^q(I, L^r) \|^{p-1} \qquad (*)$$

by Hölder in space and in time

$$\frac{1}{\bar{r}_1} + \frac{p}{r} = 1$$

$$\frac{1}{\bar{q}_1} + \frac{p}{q} = 1$$

$\therefore$

$$\frac{n}{r_1} + \frac{2}{q_1} + \left(\frac{n}{r} + \frac{2}{q} \right) p = n + 2$$

$$(p+1)\, n/2 = n+2 \qquad \text{i.e.} \qquad p = 1 + 4/n .$$

Choice of r, r_1: $\quad r = r_1 = p+1 = q = q_1 = 2 + 4/n$

Denote $p + 1 = r_0$. Develop the estimates :

$$\left.\begin{array}{l}\| U(\cdot)u_+ ; L^\infty(I, L^2) \| \\ \\ \| U(\cdot)u_+ ; L^{r_0}(I, L^{r_0}) \|\end{array}\right\} \leq c'_{r_0} \|u_+\|_2$$

$$\left.\begin{array}{l}\| F(u) ; L^\infty(I, L^2) \| \\ \\ \| F(u) ; L^{r_0}(I, L^{r_0}) \|\end{array}\right\} \leq \bar{c} \; \| u ; L^{r_0}(I, L^{r_0}) \|^p$$

So that

$$\| A(u) ; X_{r_0}(I) \| \leq c'_{r_0} \|u_+\|_2 + \bar{c} \; \| u ; L^{r_0}(I, L^{r_0}) \|^p$$

$$\| A(u) ; L^{r_0}(I, L^{r_0}) \| \leq \| U(\cdot)u_+ ; L^{r_0}(I, L^{r_0}) \| + \bar{c} \; \| u ; L^{r_0}(I, L^{r_0}) \|^p$$

Furthermore

$$\| A(u_1) - A(u_2) ; X_{r_0}(I) \| \leq \bar{c} \max_{j=1,2} \| u_j ; L^{r_0}(I, L^{r_0}) \|^{p-1} \; \| u_1 - u_2 ; L^{r_0}(I, L^{r_0}) \| .$$

Def: $B(I, R, R_0) = \left\{ u \in X_{r_0}(I) : \| u ; X_{r_0}(I) \| \leq R , \| u ; L^{r_0}(I, L^{r_0}) \| \leq R_0 \right\}$

$B(I, R, R_0)$ is a closed subset of $X_{r_0}(I)$.

Choose (for fixed u_+): $\quad c'_{r_0} \|u_+\|_2 \leq \frac{R}{2}$, $\| U(\cdot)u_+ ; L^{r_0}(I, L^{r_0}) \| \leq \frac{R_0}{2}$,

$$\bar{c} \, R_0^{p-1} \leq 1/2 , \qquad R_0 \leq R .$$

The smallness condition of R_0 is satisfied by taking

T sufficiently large.

Let $u \in B(I, R, R_0)$. Then

$$\| A(u); X_{10}(I) \| \leq R \tfrac{1}{2} + \bar{c} R_0^p \leq R$$

$$\| A(u); L^{10}(I, L^{10}) \| \leq R_0 \tfrac{1}{2} + \bar{c} R_0^p \leq R_0$$

Let $u_1, u_2 \in B(I, R, R_0)$. Then

$$\| A(u_1) - A(u_2); X_{10}(I) \| \leq \tfrac{1}{2} \| u_1 - u_2; L^{10}(I, L^{10}) \|$$

$\therefore$ A has a unique fixed point in $B(I, R, R_0)$ for T sufficiently large.

From Strichartz estimates it is obvious that $u \in L^q(I, L^r)$ for any (q, r) admissible.

Continuity in u_+: Let u_+ and u_+' satisfy

$$c_{10}' \| u_+^{(\prime)} \|_2 \leq \frac{R}{2}, \quad \| U(\cdot) u_+^{(\prime)}; L^{10}(I, L^{10}) \| \leq \frac{R_0}{2}$$

$$\text{and} \quad \bar{c} R_0^{p-1} \leq \tfrac{1}{2}, \qquad R_0 \leq R$$

Let u, u' be solutions of

$$u(t) = U(t) u_+ + (Fu)(t)$$

$$u'(t) = U(t) u_+' + (Fu')(t)$$

in $B(I, R, R_0)$. Then from

$$u(t) - u'(t) = U(t)(u_+ - u_+') + Fu(t) - Fu'(t)$$

it follows

$$\| u - u' ; X_{r_0}(I) \| \leq c'_{r_0} \| u_+ - u'_+ \|_2 + \frac{1}{2} \| u - u' ; X_{r_0}(I) \|$$

$$\therefore \quad \| u - u' ; X_{r_0}(I) \| \leq 2 c'_{r_0} \| u_+ - u'_+ \|_2$$

Using again Strichartz estimates we obtain for any (q,r) admissible

$$\| u - u' ; X_r(I) \| \leq 2 \, c''_r \, \| u_+ - u'_+ \|_2 .$$

(2) From the (IE_∞) we rewrite :

$$U(-t)u(t) - u_+ = -i \int_\infty^t dt' \; U(-t') \, f(u(t'))$$

so that

$$\| U(-t)u(t) - u_+ \|_2 = \left\| \int_\infty^t dt' \; U(t-t') \; f(u(t')) \right\|_2$$

$$\leq \left\| \int_\infty^t dt' \; U(t-t') \, f(u(t')) ; \; L^\infty([t,\infty) ; L^2) \right\|$$

qui par Strichartz et la majoration precedente $(*)$ donne

$$\ldots \leq c' \, \| u ; L^{r_0}([t,\infty), L^{r_0}) \|^p \longrightarrow 0 \quad \text{when} \quad t \to \infty .$$

$\square$

Proposition IVP_∞ in L^2 for HE. Let $V \in L^s(\mathbb{R}^n)$, $n \geq 2$, $s = \frac{n}{2}$.

Then the statements and conclusions of the Proposition "IVP_∞ in L^2 for NLS" hold with the following modifications:

$$f(u) = (V * |u|^2)\, u$$

$X_{p+1}(I)$ is replaced by $X_{r_0}(I)$ with $r_0 = \frac{2n}{n-1} \longleftrightarrow \delta(r_0) = 1/2$

Proof. The estimate $(*)$ in the Proposition "IVP_∞ in L^2 for NLS" is replaced by

$$\| F(u_1) - F(u_2); L^{q_2}(I, L^{r_2}) \| \leq \bar{C} \, \|V\|_s \, \| u_1 - u_2; L^q(I, L^r) \| \, \underset{j=1,2}{\mathrm{Max}} \| u_j; L^q(I, L^r) \|^2$$

with the Hölder conditions

$$\frac{1}{r_1} + \frac{3}{r} + \frac{1}{s} = 2$$

$$\frac{1}{q_1} + \frac{3}{q} = 1$$

$$\therefore \qquad \frac{n}{s} = 2$$

Choice of $\quad r = r_1 = r_0 \quad \longrightarrow \quad \frac{n}{s} = 4\,\delta(r_0)$

$$\longrightarrow \quad \delta(r_0) = \frac{1}{2} \longleftrightarrow r_0 = \frac{2n}{n-1}$$

The rest of the proof is exactly the same, with the replacement of the exponent $p-1$ by 2.

$\square$

Proposition IVP$_\infty$ in H^1 for (NLS). Let $p \geq 1 + 4/n$ and $p-1 < 4/(n-2)$ for $n \geq 3$.

Then: (1) For any $u_+ \in H^1$, there exists $T = T(u_+)$ such that the (IE$_\infty$)

$$u(t) = U(t)u_+ - i\int_\infty^t dt'\, U(t-t')\, f(u(t'))$$

with $f(u) = \lambda |u|^{p-1}u$, $\lambda \in \mathbb{C}$

has a unique solution $u \in X^1_{p+1}(I)$ where $I = [T, \infty)$. Furthermore,

for all (q,r) admissible, $u, \nabla u \in L^q(I, L^r)$ and $(u, \nabla u)$ are

continuous functions of $u_+ \in H^1$ with values in $L^q(I, L^r) \times L^q(I, L^r)$.

(2) The solution u admits u_+ as asymptotic state in H^1, i.e.

$$\| U(-t)u(t) - u_+ ; H^1 \| = \| u(t) - U(t)u_+ ; H^1 \| \longrightarrow 0 \text{ when } t \to \infty.$$

Proof (1) Def:
$$F(u) \equiv -i\int_\infty^t dt'\, U(t-t')\, f(u(t'))$$

$$A(u)(t) \equiv U(t)u_+ + F(u)(t)$$

We want to solve the equation

$$u = Au$$

From the following estimates we will see that the integral in F is well defined. It will be seen that, for $T \leq s, t$,

$$\left\| \int_s^t dt'\, U(t-t')\, f(u(t')) ; L^\infty((T,\infty); H^1) \right\| \longrightarrow 0 \text{ when } T \to \infty.$$

Besides estimates for f we need estimates for ∇f. We will use the formulae

$$\nabla(|u|^{p-1}u) = |u|^{p-1}\nabla u + u\,\nabla|u|^{p-1} = |u|^{p-1}\nabla u + (p-1)\,|u|^{p-2}u\,\mathrm{Re}\left(\frac{\bar{u}}{|u|}\nabla u\right) \quad (0)$$

so that

$$\left|\nabla |u|^{p-1}u\right| \leq p\,|u|^{p-1}\,|\nabla u| \qquad (0')$$

(See S. Agmon: Lectures on exponential decay of solutions of second order elliptic equations, Math. Notes, Princeton Univ. Press

Leinfelder – Simader)

In the same way as in (*) of "Proposition IVP$_\infty$ in L^2 for NLS" we estimate ((q,r), (q_1,r_1), (q_2,r_2) admissible)

$$\left\| F(u_1)-F(u_2);\ L^{q_2}(I,L^{r_2})\right\| \leq c\,\left\| f(u_1)-f(u_2);\ L^{\bar{q}_1}(I,L^{\bar{r}_1})\right\|$$

$$\leq \bar{c}\,\left\| u_1-u_2;\ L^{q}(I,L^{r})\right\| \max_{j=1,2}\left\| u_j;\ L^{q_3}(I,L^{m})\right\|^{p-1}$$

and similarly

$$\left\|\nabla F(u);\ L^{q_2}(I,L^{r_2})\right\| \leq c\,\left\| |u|^{p-1}|\nabla u|;\ L^{\bar{q}_1}(I,L^{\bar{r}_1})\right\|$$

$$\leq \bar{c}\,\left\| \nabla u;\ L^{q}(I,L^{r})\right\| \left\| u;\ L^{q_3}(I,L^{m})\right\|^{p-1}$$

with $I=[T,\infty)$ and Hölder in space and time

$$\frac{1}{r_1}+\frac{1}{r}+\frac{p-1}{m}=1 \qquad (1)$$

$$\frac{1}{q_1}+\frac{1}{q}+\frac{p-1}{q_3}=1 \qquad (2)$$

$\therefore$
$$\frac{n}{r_1}+\frac{2}{q_1}+\frac{n}{r}+\frac{2}{q}+p-1\left(\frac{n}{m}+\frac{2}{q_3}\right)=n+2$$

$$(p-1)\left(\frac{n}{m}+\frac{2}{q_3}\right)=2$$

i.e.
$$(p-1)\left(\frac{n}{2}-\left(\delta(m)-\frac{2}{q_3}\right)\right)=2 \qquad (3)$$

We aim at achieving the contraction by the use of a term of the type $\int_T^\infty dt\, \|u\|_{r_3}^{q_3} < \infty$ with (q_3, r_3) admissible, $q_3 < \infty$,

so that $\int_{t_1}^{t_2} \|u\|_{r_3}^{q_3} dt \longrightarrow 0$ when $t_1, t_2 \longrightarrow \infty$.

So we want $\boxed{q_3 < \infty}$.

Rewrite (3) as $(p-1)\left(\frac{n}{2}-\rho\right)=2$

with $\rho = \delta(m)-\delta(r_3) = \delta(m) - \frac{2}{q_3}$

Choice of the parameters: $r = r_1 = (p+1) \equiv r_0$. From (1) it follows that $m = p+1 = r_0$. Denote $\delta_0 \equiv \delta(r_0)$.

So $\rho = \delta_0 - \delta(r_3)$.

The assumption $p-1 \geq 4/n$ is equivalent to $\rho \geq 0$

or to $\delta(r_3) \leq \delta_0$. Recall that, by Strichartz,

for $n=1$ $\delta_0 \leq 1/2$, for $n=2$ $\delta_0 < 1$, for $n \geq 3$ $\delta_0 \leq 1$.

Therefore for $n=1$ $\rho < 1/2$ and for $n \geq 2$ $\rho < 1$

(do not forget $\delta(r_3) = \frac{2}{q_3} > 0$). This corresponds for

$n = 1,2$ to the non existence of upper bounds on p and

for $n \geq 3$ to the restriction

$$p - 1 < \frac{2}{\frac{n}{2} - 1}$$

Now we perform the estimate:

$$\|u; L^{q_3}(L^{m})\|^{p-1} = \|u; L^{q_3}(L^{r_3 p})\|^{p-1} \leq$$

$$\leq C \|u; L^{q_3}(L^{r_3})\|^{(1-\rho)(p-1)} \|\nabla u; L^{q_3}(L^{r_3})\|^{\rho(p-1)}$$

$\rho = \delta_0 - \delta(r_3)$. Denote $\sigma_0 = (1-\rho)(p-1)$

We need $\delta(r_3)$. From (2) we have

$$(p-1)\delta(r_3) = 2(1-\delta_0)$$

(recall $\frac{n}{2}(p-1) = (p+1)\delta_0$)

$$\therefore \quad \sigma_0 = (1 - \delta_0 + \delta(r_3))(p-1) = (p-1)(1-\delta_0) + 2(1-\delta_0)$$

$$= (p+1)(1-\delta_0) = (p+1) - \frac{n}{2}(p-1)$$

Important $\sigma_0 > 0$.

The previous estimate can be rewritten :

$$\|u; L^{q_3}(L^{r_3})\|^{p-1} \leq c \; \| u; L^{q_3}(L^{r_3})\|^{\sigma_0} \; \| \nabla u; L^{q_3}(L^{r_3})\|^{p-1-\sigma_0}$$

$$\leq c \; \| u; L^{q_0}(L^{r_0})\|^{\sigma_0\mu_0} \; \| u; L^{\infty}(L^{2})\|^{\sigma_0(1-\mu_0)} \; \| \nabla u; L^{q_3}(L^{r_3})\|^{p-1-\sigma_0}$$

with $\quad \mu_0 = \frac{q_0}{q_3} = \frac{\delta(r_3)}{\delta_0} = \frac{2}{p-1} \quad \frac{1-\delta_0}{\delta_0}$

so that $\quad \sigma_0 \mu_0 = 2 \; \frac{p+1}{p-1} \; \frac{(1-\delta_0)^2}{\delta_0}$

Better estimate (more direct) :

$$\|u; L^{q_3}(L^{r_0})\|^{p-1} \leq c \; \| u; L^{q_0}(L^{r_0})\|^{\sigma} \; \| u; L^{\infty}(L^{r_0})\|^{p-1-\sigma}$$

$$\sigma = (p-1) \frac{q_0}{q_3} = \frac{2(1-\delta_0)}{\delta_0} > 0$$

$$\leq c \; \| u; L^{q_0}(L^{r_0})\|^{\sigma} \; \| u; L^{\infty}(H^1)\|^{p-1-\sigma}$$

$p-1-\sigma = \frac{1}{\delta_0}\{(p-1)\delta_0 + 2\delta_0 - 2\} = \frac{1}{\delta_0}\{(p+1)\delta_0 - 2\} = \frac{1}{\delta_0}\left\{\frac{n}{2}(p-1) - 2\right\} \geq 0$

Reproduction and contraction estimates :

$$\| A(u); \overline{X}^1_{r_0}(I)\| \leq c'_{r_0} \|u_+; H^1\| + \bar{c} \| u; \overline{X}^1_{r_0}(I)\| \; \| u; L^{\infty}(I, H^1)\|^{p-1-\sigma} \; \| u; L^{q_0}(I, L^{r_0})\|^{\sigma}$$

$$\| A(u); L^{q_0}(I, L^{r_0})\| \leq \| U(\cdot)u_+; L^{q_0}(I, L^{r_0})\| + \bar{c} \; \| u; L^{q_0}(I, L^{r_0})\|^{1+\sigma} \; \| u; L^{\infty}(I, H^1)\|^{p-1-\sigma}$$

where

$$\overline{X}_r(I) = \{ u \in L^\infty(I, L^2) \cap L^q(I, L^2) \text{ with } (q,r) \text{ adm...} \}$$

$$\overline{X}_r^1(I) = \{ u \in \overline{X}_r(I), \quad \nabla u \in \overline{X}_r(I) \}$$

The norms for those Banach spaces are the same as those for $X_r^1(I)$. The difference is that we do not require $u \in \mathcal{C}(I, L^2)$ or/and $\nabla u \in \mathcal{C}(I, L^2)$.

This has some technical reasons needed to prove that the ball in which we will contract is <u>closed</u>.

$$\| A(u_1) - A(u_2) ; X_{r_0}(I) \| \leq \bar{C} \| u_1 - u_2 ; \bar{X}_{r_0}(I) \| \max_{j=1,2} \| u_j ; L^\infty(I, H^1) \|^{p-1-\sigma} \| u_j ; L^{q_0}(I, L^{r_0}) \|^\sigma$$

Def : $\bar{B}(I, R, R_0) = \left\{ u \in \bar{X}^1_{r_0}(I) : \| u ; \bar{X}^1_{r_0}(I) \| \leq R, \quad \| u ; L^{q_0}(I, L^{r_0}) \| \leq R_0 \right\}$

- It will be proved later that $B(I, R, R_0)$ is a closed subset of $X_{r_0}(I)$

Choose (for fixed u_+) : $\quad C'_{r_0} \| u_+ ; H^1 \| \leq \frac{R}{2}$ $\quad (*)$

$$\| U(\cdot) u_+ ; L^{q_0}(I, L^{r_0}) \| \leq \frac{R_0}{2} \quad (**)$$

$$\bar{C} R^{p-1-\sigma} R_0^\sigma \leq 1/2 \quad (***)$$

and $R_0 \leq R$.

Let $u \in \bar{B}(I, R, R_0)$. Then

$$\| A(u) ; \bar{X}^1_{r_0}(I) \| \leq \frac{R}{2} + \bar{C} R_0^\sigma R^{p-\sigma} \leq R$$

$$\| A(u) ; L^{q_0}(I, L^{r_0}) \| \leq \frac{R_0}{2} + \bar{C} R_0^{\sigma+1} R^{p-1-\sigma} \leq R_0$$

Let $u_1, u_2 \in B(I, R, R_0)$. Then

$$\| A(u_1) - A(u_2) ; \bar{X}_{r_0}(I) \| \leq \frac{1}{2} \| u_1 - u_2 ; X_{r_0}(I) \|$$

$\therefore$ A has a ! fixed point in $\bar{B}(I, R, R_0)$ for T sufficiently large, since the smallness condition of R_0 is satisfied by taking T sufficiently large.

From Strichartz estimates it is obvious that $u, \nabla u \in L^q(I, L^r)$

for any (q,r) admissible. Again Strichartz provides the continuity in time of the solutions with values in H^1.
Continuity in u_+: let u_+, u'_+, R, R_0 satisfy $(*), (**), (***)$

and $R_0 \leq R$. Let u, u' be solutions of

$$u(t) = U(t)u_+ + (Fu)(t)$$

$$u'(t) = U(t)u'_+ + (Fu')(t)$$

in $\bar{B}(I, R, R_0)$. Then from

$$u(t) - u'(t) = U(t)(u_+ - u'_+) + (Fu)(t) - (Fu')(t)$$

it follows

$$\| u - u' ; \bar{X}_{r_0}(I) \| \leq c'_{r_0} \| u_+ - u'_+ ; L^2 \| + \tfrac{1}{2} \| u - u' ; \bar{X}_{r_0}(I) \|$$

$$\| u - u' ; \bar{X}_{r_0}(I) \| \leq 2c'_{r_0} \| u_+ - u'_+ ; L^2 \| \qquad (4)$$

By Strichartz we obtain, for any admissible (q, r)

$$\| u - u' ; \bar{X}_r(I) \| \leq 2c'_{r_0} \| u_+ - u'_+ ; L^2 \| \qquad (5)$$

passing through:

$$\| u - u' ; \bar{X}_r(I) \| \leq c'_{r_0} \| u_+ - u'_+ ; L^2 \| + \tfrac{1}{2} \| u - u' ; \bar{X}_{r_0}(I) \| .$$

and using (4).

In order to have the continuity in $\bar{X}^1_2(I)$ we estimate

$$\nabla(u-u') = U(t)\,(u_+ - u'_+) - i\int_t^\infty dt'\,U(t-t')\,\nabla\left(|u|^{p-1}u - |u'|^{p-1}u'\right)$$

in $X_{r_0}(I)$. By ( ) pg. 8 we can rewrite

$$\nabla\left(|u|^{p-1}u - |u'|^{p-1}u'\right) = \phi + \psi$$

$$\phi \equiv |u'|^{p-1}\nabla(u-u') + (p-1)|u'|^{p-2}u'\,\mathrm{Re}\left(\frac{\bar{u}'}{|u'|}\cdot\nabla(u-u')\right)$$

$$\psi \equiv \left(|u|^{p-1} - |u'|^{p-1}\right)\nabla u + (p-1)\left(|u|^{p-2}u\,\mathrm{Re}\left(\frac{\bar{u}}{|u|}\cdot\nabla u\right) - |u'|^{p-2}u'\,\mathrm{Re}\,\frac{\bar{u}'}{|u'|}\cdot\nabla u\right).$$

By using (see (0') pg 8)

$$|\phi| \le p\,|u'|^{p-1}\,|\nabla(u-u')|$$

and the previous estimates, in particular the estimates of

$\|\nabla F(u);\, L^{q_2}(I, L^{r_2})\|$ we obtain

$$\|\nabla(u-u');\bar{X}_{r_0}(I)\| \le c'_{r_0}\|u_+ - u'_+;\dot{H}^1\| + \frac{1}{2}\|\nabla(u-u');\bar{X}_{r_0}(I)\| + c\,\|\psi;\, L^{\bar{q}_1}(I, L^{\bar{r}_1})\|$$

so that

$$\|\nabla(u-u');\bar{X}_{r_0}(I)\| \le 2c'_{r_0}\|u_+ - u'_+;\dot{H}^1\| + 2c\,\|\psi;\, L^{\bar{q}_1}(I, L^{\bar{r}_1})\|.$$

Let now $u' = u^{(n)}$ and $u'_+ = u^{(n)}_+$ with $u^{(n)}_+ \longrightarrow u_+$ in $\dot{H}^1$.

Then, by ( ), $X_{r_0}(I)\text{-}\lim u^{(n)} = u$.

Let us denote now by ψ_n the function ψ where u'

is replaced by $u^{(n)}$. Let us suppose that $\|\varphi_n; L^{\bar q_1}(I, L^{\bar r_1})\|$ does not converge to 0 when $n \to \infty$. This implies that $\exists\, \varepsilon > 0$ and a subsequence of $\{u_n\}$, still denoted in the same way, such that

$$\|\varphi_n; L^{\bar q_1}(I, L^{\bar r_1})\| \geq \varepsilon \qquad \forall n \qquad \otimes$$

From ( ), possibly passing to another subsequence, still denoted with the same notation, we have

$$u^{(n)} \longrightarrow u \qquad \text{a.e. in } I \times \mathbb{R}^3 \qquad \boxtimes$$

$$|u^{(n)}| \leq v \qquad \text{for some } v \in X_{r_0}(I).$$

On the other hand, by the estimates already performed,

$$|\varphi_n| \leq (p-1)\left(|u|^{p-1} + |v|^{p-1}\right)|\nabla u| \in L^{\bar q_0}(I, L^{\bar r_0})$$

(remember $q_0 = q_1$ $r_0 = r_1$).

Since $\varphi_n \longrightarrow 0$ a.e in $I \times \mathbb{R}^3$ by $\boxtimes$, Lebesgue's dominated convergence theorem implies $\|\varphi_n; L^{\bar q_0}(I, L^{\bar r_0})\| \to 0$ in contradiction with $\otimes$. This proves that $\|\nabla(u - u^{(n)}); \bar X_{r_0}(I)\| \longrightarrow 0$.

Applying Strichartz again we obtain for any (q, r) admissible $\|\nabla(u - u^{(n)}); \bar X_r(I)\| \longrightarrow 0$ when $n \to \infty$.

Proof that $\bar{B}(I,R,R_0) \subset \bar{X}^1_{r_0}(I)$ is closed in $\bar{X}_{r_0}(I)$.

Let $\{u_j\}$ be a sequence $u_j \in \bar{B}(I,R,R_0)$ such that

$$u_j \longrightarrow u \quad \text{in } \bar{X}_1(I).$$

This means $\quad u_j \longrightarrow u \quad$ in $\quad L^\infty(I,L^2)$ and $L^{q_0}(I,L^{r_0})$

and that the sequence ∇u_j be bounded in $L^\infty(I,L^2) \cap L^{q_0}(I,L^{r_0})$.

If $\phi \in \mathcal{C}^\infty_0(I,\mathbb{R}^n)$, by the definition of ∇u_j, we have

$$\langle \nabla u_j, \phi \rangle = - \langle u_j, \nabla\phi \rangle$$

so that
$$\lim_{j\to\infty} \langle \nabla u_j, \phi \rangle = - \langle u, \nabla\phi \rangle \qquad \Delta$$

On the other hand $\quad L^\infty(I,L^2) = (L^1(I,L^2))^*$ and $L^{q_0}(I,L^{r_0}) = (L^{\bar{q}_0}(I,L^{\bar{r}_0}))^*$

where $\quad \frac{1}{q_0} + \frac{1}{\bar{q}_0} = 1, \quad \frac{1}{r_0} + \frac{1}{\bar{r}_0} = 1.$

$L^1(I,L^2)$ and $L^{\bar{q}_0}(I,L^{\bar{r}_0})$ are separable so that balls in $L^\infty(I,L^2)$ and $L^{q_0}(I,L^{r_0})$ are metrizable with respect to the w^*-topology. Therefore the well-known w^*-compactness of balls implies w^* sequential compactness.

Therefore there exists a subsequence of $\{u_j\}$, $\{u_{j_k}\}$ such that $\vec{\nabla} u_{j_k}$ converges when $k \longrightarrow \infty$

w^* in $L^\infty(I,L^2)$ to $\vec{v}_1 \in L^\infty(I,L^2)$ and to $\vec{v}_2 \in L^{q_0}(I,L^{r_0})$ w^* in $L^{q_0}(I,L^{r_0})$. We write

$$\lim_{k\to\infty} \langle \vec{\nabla} u_{j_k}, \varphi_1 \rangle = \langle \vec{v}_1, \varphi_1 \rangle$$

$\forall \ \varphi_1 \in L^1(I,L^2)$, and

$$\lim_{k\to\infty} \langle \vec{\nabla} u_{j_k}, \varphi_2 \rangle = \langle \vec{v}_2, \varphi_2 \rangle$$

$\forall \ \varphi_2 \in L^{\bar{q}_0}(I,L^{\bar{r}_0})$. Since $\mathcal{C}_0^\infty(I\times\mathbb{R}^n)$ is dense both in $L^1(I,L^2)$ and $L^{\bar{q}_0}(I,L^{\bar{r}_0})$ we can identify the limits $\vec{v}_1, \vec{v}_2$ through Δ:

$$\vec{v}_1 = \vec{v}_2 = \vec{\nabla} u$$

Conclusion: $(u, \nabla u) \in \overline{X}_{r_0}(I)$, i.e. $u \in \overline{X}^1_{r_0}(I)$.

Since the w^* limit preserves the bounds, we have

$u \in \overline{B}(I, R, R_0)$.

(2) From the (IE$_\infty$) we rewrite

$$U(-t)u(t) - u_+ = -i \int_\infty^t dt' \, U(-t') \, f(u(t'))$$

so that

$$\| U(-t)u(t) - u_+ ; H^1 \| = \left\| \int_\infty^t dt' \, U(t-t') \, f(u(t')) ; H^1 \right\|$$

$$\leq \left\| \int_\infty^t dt' \, U(t-t') \, f(u(t')) ; L^\infty([t,\infty) ; H^1) \right\| \leq$$

$$\leq C \, \| u ; X^1_{r_0}([t,\infty)) \| \, \| u ; L^{q_3}([t,\infty) ; L^{r_1}) \|^{p-1}$$

$$\leq C \, \| u ; X^1_{r_0}([t,\infty)) \| \, \| u ; L^\infty([t,\infty), H^1) \|^{p-1-\sigma} \, \| u ; L^{q_0}([t,\infty) ; L^{r_0}) \|^{\sigma}$$

The first two terms are bounded, the third one $\longrightarrow 0$

when $t \to \infty$ since $\sigma > 0$.

$\square$

Proposition IVP$_\infty$ in H^1 for HE. Let $n \geq 2$, let V complex function $\in L^s_1$, with $1 \leq s$ and $2 \leq \frac{n}{s} < 4$. Let $r_0 = \frac{2n}{n-1}$ (so that $\delta(r_0) = \frac{1}{2}$). Let $u_+ \in H^1$. Then

(1) For any $u_+ \in H^1$, there exists $T = T(u_+)$ such that the (IE$_\infty$)

$$u(t) = U(t)u_+ - i \int_\infty^t dt' \, U(t-t') \, f(u(t'))$$

with $f(u) = (V * |u|^2)u$ — space convolution

has a unique solution $u \in X^1_{r_0}(I)$ where $I = [T, \infty)$. Furthermore $u \in \bigcap_{\forall (q,r) \text{ admissible}} L^q(I, L^r) \equiv X^1(I)$ and is a continuous function from $u_+ \in H^1$ to $X^1(I)$.

(2) The solution u admits u_+ as asymptotic state in H^1, i.e.

$$\| U(-t)u(t) - u_+ ; H^1 \| = \| u(t) - U(t)u_+ ; H^1 \| \longrightarrow 0 \quad \text{when } t \longrightarrow \infty$$

Proof. Is similar to that for NLS to which we refer for details

Let $I = [T, \infty)$ and $(q_2, r_2), (q, r), (q_1, r_1)$ be admissible pairs

In the same way as for NLS we have

$$\| F(u_1) - F(u_2); L^{\bar{q}_2}(I, L^{\bar{r}_2}) \| \leq \bar{c} \|V\|_s \| u_1 - u_2 ; L^q(I, L^r) \| \operatorname*{Max}_{j=1,2} \| u_j ; L^{q_3}(L^m) \|^2$$

$$\|\nabla F(u); L^{q_2}(I, L^{r_2})\| \leq c\, \|V\|_s \, \|\nabla u; L^{q}(I, L^{r})\| \, \|u; L^{q_3}(I, L^{m})\|^2$$

with

$$\frac{1}{s} + \frac{1}{r_1} + \frac{1}{r} + \frac{2}{m} = 2 \qquad \longleftrightarrow \qquad \frac{n}{s} = \delta(r_1) + \delta(r) + 2\delta(m)$$

$$\frac{1}{q_1} + \frac{1}{q} + \frac{2}{q_3} = 1$$

From the two previous equality, using admissibility for (q_1, r_1), (q, r):

$$\frac{n}{s} = 2 + 2\left(\delta(m) - \frac{2}{q_3}\right)$$

Choose: $r = r_1 = r_0$ so that $q = q_1 = q_0 = 4$ $\left(\delta(r_0) = \frac{1}{2}\right)$

Then $q_3 = 4$ and $\frac{n}{s} = 1 + 2\delta(m)$

Taking $\frac{1}{2} \leq \delta(m) < 3/2$ we sweep the whole interval of allowed values of $\frac{n}{s}$. (Remember for $n=2$ $\delta(m) \leq 1$)

We estimate by Holder - Sobolev

$$\|u; L^{q_3}(I, L^{m})\| \leq \|u; L^{q_3}(I, L^{r_3})\|^{1-\rho} \, \|\nabla u; L^{q_3}(I, L^{r_3})\|^{\rho}$$

with $\rho = \delta(m) - \delta(r_3)$ and $0 \leq \rho \leq 1$. Rewrite it:

$$\|u; L^{4}(I, L^{m})\| \leq \|u; L^{4}(I, L^{r_0})\|^{1-\rho} \, \|\nabla u; L^{4}(I, L^{r_0})\|^{\rho} \qquad (*)$$

with $\rho = \delta(m) - \frac{1}{2}$.

Let $\delta(m) < \frac{3}{2}$. Then $1-\rho>0$. Therefore the first factor at the RHS of (*) converges to zero when $T\to\infty$ and provides the contraction factor since

$$\| \nabla u ; L^4(I, L^{10}) \| \leq \| u ; X^1_{10}(I) \| .$$

(The same way as for NLS).

If $\delta(m) = \frac{3}{2}$ we are in the limiting case $\rho = 1$

The rest of the proof follows the same pattern as that for NLS, with the simplification that we can do the contraction in the Balls of X^1_{10} since there is no problem with the regularity of $f(u)$ in the variable u

$\square$