The time decay in the definition of $X_{p+1}([T,\infty))$ or $X^1_{p+1}([T,\infty))$.

i.e. $u \in L^q([T,\infty); L^r)$, with (q,r) compatible is not the optimal available for $u(t) = U(t)u_0$. In fact

$$\| u(t) = U(t)u_0 \|_r \leq C\,|t|^{-\delta(r)} \|u_0\|_{\bar r}$$

where $2 \leq r \leq \infty$ and $\frac{1}{r} + \frac{1}{\bar r} = 1$, for $u_0 \in L^{\bar r}$

(Check it on gaussian u_0, where an explicite computation can be done)

But the spaces L^r, $L^{\bar r}$ are not preserved by the evolution $U(t)$. Convenient spaces are introduced by the use of the operator

$$J(t) = x + it\nabla$$

This operator is the infinitesimal generator of the Galilei transformations: for any function of time-space u define, for any vector v (velocity) def

$$\begin{cases} x \longrightarrow x - vt \\ t \longrightarrow t \end{cases}$$

galilei transport of space time

$$(\mathcal{G}_v u)(t,x) = \exp\left\{i\left(x\cdot v - \frac{1}{2}v^2 t\right)\right\} u(t, x - vt)$$

$$\therefore \qquad (\nabla_v \mathcal{G}_v u)(t,x)\Big|_{v=0} = i\, J(t)\, u(t,x)$$

Some of the properties of $J(t)$:

a) $$J(t)\, U(t-t') = U(t-t')\, J(t') \qquad \otimes$$

In particular :

$$J(t)\, U(t) = U(t)\, x$$

Useful in :

$$J(t) \int_{t_0}^{t} dt'\, U(t-t')\, f(u(t')) = \int_{t_0}^{t} dt'\, U(t-t')\, J(t')\, f(u(t'))$$

Def $M(t) = \exp \dfrac{i x^2}{2t}$ (multiplication by ...) $t \neq 0$

b) $$M(t)\, it\nabla\, M(-t) = J(t)$$

Useful in : From Sobolev in $L(\mathbb{R}^n)$: $\| v \|_r \leq C_r \| v \|_2^{1-\delta(r)} \| \nabla v \|_2^{\delta(r)}$

for $\delta(r) \leq 1$ if $n \geq 3$

$\delta(r) < 1$ if $n = 2$

$\delta(r) \leq \frac{1}{2}$ if $n = 1$

Apply the previous inequality to $M(-t)u(t)$:

$$\|u(t)\|_r \leq C_r \|u(t)\|_2^{1-\delta(r)} \|\nabla M(-t)u(t)\|_2^{\delta(r)}$$

$$= C_r |t|^{-\delta(r)} \|u(t)\|_2^{1-\delta(r)} \|J(t)u(t)\|_2^{\delta(r)}$$ ⊠

∴ If we use a space where $\|J(\cdot)u(\cdot)\|_2 \in L^\infty([T,\infty); L^2)$

the $u(t)$ has the optimal decay.

This suggest the definition of the spaces.

$$\Sigma = \{u \in H^1(\mathbb{R}^n),\ xu \in L^2(\mathbb{R}^n)\} \quad (\text{replacement of } H^1)$$

and

$$Y_r^1(I) = \{u \in \mathcal{C}(I,\Sigma);\ u(\cdot), \nabla u(\cdot), J(\cdot)u(\cdot) \in L^q(I,L^r)\ \ q,r \text{ adm.}\}$$

$$Y^1(I) = \{u \in \mathcal{C}(I,\Sigma);\ u, \nabla u, Ju \in L^q(I,L^r)\ \ \forall (q,r) \text{ admissible}\}$$

c) From $\quad it\nabla(|u|^{p-1}u) = |u|^{p-1}(it\nabla u) + (p-1)|u|^{p-2}u\, \mathrm{Im}\frac{\bar{u}}{|u|}(it\nabla u)$

$$J(t)|u|^{p-1}u = |u|^{p-1}J(t)u + (p-1)|u|^{p-2}\,\mathrm{Im}\frac{\bar{u}}{|u|}J(t)u$$

∴ $\quad \left|J(t)(|u|^{p-1}u)\right| \leq p\,|u|^{p-1}\,|J(t)u|$ ⊠

<u>Proposition IVP$_\infty$ in Σ for NLS</u>. Let $p > 1 + \frac{4}{n+2}$ for $n \geq 2$, $p > 3$ for $n = 1$ and $p - 1 < \frac{4}{(n-2)}$ for $n \geq 3$. Then

(1) For any $u_+ \in \Sigma$, there $\exists$ $T = T(u_+)$ such that the (IE$_\infty$)

$$u(t) = U(t) u_+ - i \int_\infty^t dt' \, U(t-t') f(u(t'))$$

with $f(u) = \lambda |u|^{p-1} u$ $\quad \lambda \in \mathbb{C}$

has a unique solution in $Y^1_{p+1}(I)$ with $I = [T, \infty)$. Furthermore $u \in Y^1(I)$ and u is continuous from Σ to $Y^1(I)$.

(2) The solution admits u_+ as asymptotic state in Σ, i.e.

$$\| U(-t) u(t) - u_+ ; \Sigma \| \longrightarrow 0 \quad \text{when } t \longrightarrow \infty.$$

<u>Proof</u>. The first part of the proof is the same as the same part of the proof of Proposition IVP$_\infty$.

Using a), c), precisely $\otimes$ and $\otimes$, we have, for (q, r), (q_1, r_1); (q_2, r_2) admissible:

$$\| F(u_1) - F(u_2); L^{\bar{q}_2}(I, L^{\bar{r}_2}) \| \leq \bar{C} \| u_1 - u_2 ; L^q(I, L^r) \| \max_{j=1,2} \| u_j ; L^{q_3}(I, L^m) \|^{p-1}$$

$$\| J(\cdot) F(u) ; L^{\bar{q}_2}(I, L^{\bar{r}_2}) \| \leq \bar{C} \| J(\cdot) u ; L^q(I, L^r) \| \, \| u ; L^{q_3}(I, L^m) \|^{p-1}$$

where $I = [T, \infty)$, with the following equations for the parameters (q, r), q_3, m

$$\begin{cases} (p-1)\left(\frac{n}{2}-\delta(m)\right) = 2\delta(r) & (*)_1 \\ (p-1)\left(\frac{n}{2}-\delta(m)+\frac{2}{q_3}\right) = 2 & (*)_2 \end{cases}$$

which are Eqs. (1), (2) of pag 9 (Proof of IVP_∞ in H^1 for NLS).

where we have chosen $\delta(r_1)=\delta(r)$ ($q_1=q$). Now

the treatment of $\| u; L^{q_3}(I, L^m)\|$ is different from

that in the case H^1. From b), precisely ☒, we have

$$\|u(t)\|_m \leq C_m |t|^{-\delta(m)} \|u(t)\|_2^{1-\delta(m)} \|J(t)u(t)\|_2^{\delta(m)}$$

$$\leq C |t|^{-\delta(m)} \| u; Y_2^1(I)\|$$

for $0 \leq \delta(m) \leq 1$ if $n \geq 3$

$0 \leq \delta(m) < 1$ if $n=2$

$0 \leq \delta(m) \leq \frac{1}{2}$ if $n=1$.

In order to achieve that $\lim_{T\to\infty} \int_T^\infty dt \, \|u\|_m^{q_3} = 0$

(needed for the contraction) we require $q_3\delta(m) > 1$.

So we have to look for parameters that satisfy $(*)_1$ and $(*)_2$

with the condition $\delta(m) > 1/q_3 > 0$ which is equivalent to

$$-\delta(m) < \rho \equiv \delta(m) - \frac{2}{q_3} < \delta(m)$$

Inserting the previous inequalities inside $(*)_2$ yields

$$(p-1)\left(\frac{n}{2}-\delta(m)\right) < 2 < (p-1)\left(\frac{n}{2}+\delta(m)\right) \qquad (**)$$

which provides the upper and lower strict bounds for $(p-1)$.

- If $n \geq 3$ we choose $\delta(m) = 1$, so that $\rho = 1 - \frac{2}{q_3}$

and $(**)$ becomes

$$(p-1)\left(\frac{n}{2}-1\right) < 2 < (p-1)\left(\frac{n}{2}+1\right) \qquad (***)_3$$

They are the restriction on p of the statement.

For $1 < q_3 < \infty$ $\quad -1 < \rho < 1$ so that p, determined as function of ρ spans the interval $\frac{4}{n+2} < p-1 < \frac{4}{n-2}$

- If $n = 2$ we choose $\delta(m) = 1 - \varepsilon$ so that $(**)$ becomes

$$(p-1)\varepsilon < 2 < (p-1)(2+\varepsilon) \qquad (***)_2$$

and we have the restriction on p of the statement.

$\rho = 1 - \frac{2}{q_3} -$ and p spans the allowed interval.

- If $n = 1$ we choose $\delta(m) = \frac{1}{2}$ so that $(**)$ becomes

$$0 < 2 < p-1$$

i.e. the restrictions on p. As previously spans the allowed since $p-1 = \frac{2}{\frac{1}{2}-\rho}$ and $-1/2 < \rho < \frac{1}{2}$.

In the case $n \geq 3$, the existence of $\delta(r) < 1$ which satisfies $(*)_1$ is a consequence of the left inequality of $(***)$.

In the case $n=2$ the left inequality of $(***)_2$ gives

$$(p-1)\varepsilon = 2\delta(r)$$

In the case $n=1$ $(*)_1$ is satisfied with $\delta(r)=0$ (i.e. $r=2$)

and $p-1 = q_3$ with $q_3 > 2$.

Finally we have to check that

$$(p-1)\left(\frac{n}{2} - \delta(m)\right) \leq 2\delta(p+1)$$

since we state the $\exists!$ in $Y^1_{(p+1)}$.

Case $n \geq 3$: $\frac{n}{2} - 1 \leq 2\,\frac{\delta(p+1)}{p-1} = \frac{n}{p+1}$

$$\longleftrightarrow \quad (p+1)\left(\frac{n}{2}-1\right) = (p-1)\left(\frac{n}{2}-1\right) + (n-2) \leq n$$

which is satisfies from $(p-1)\left(\frac{n}{2}-1\right) < 2$

Case $n=2$ $(p-1)\varepsilon < 2\delta(p+1) \longleftrightarrow \varepsilon < \frac{2}{p+1}$ O.K.

by choosing ε small enough

Case $n=1$ O.K. since $\delta(r)=0$.

□

Proposition IVP_∞ in Σ for HE. Let $n \geq 2$, let V complex function $\in L^s$, with $s \geq 1$ and $1 < \frac{n}{s} < 4$. Let r_0 such that $\delta(r_0) = Max\left(\frac{n}{4s}, \frac{n}{2s} - \frac{1}{2}\right)$. Let $u_+ \in \Sigma$. Then

(1) For any $u_+ \in \Sigma$, there $\exists$ $T = T(u_+)$ such that the IE_∞ with $f(u) = (V * |u|^2)u$ has a unique solution $u \in Y^1_{r_0}(I)$ where $I = [T, \infty)$. Furthermore $u \in Y^1(I)$ and is a continuous function of $u_+ \ni H^1$ with values in $Y^1(I)$.

(2) The solution u admits u_+ as asymptotic state in Σ, i.e.

$$\| U(-t)u(t) - u_+ ; \Sigma \| \longrightarrow 0 \quad \text{when } t \to \infty$$

Proof. The proof uses elements from IVP_∞ in Σ for NLS and IVP_∞ in H^1 for HE. After the usual estimates it remains to prove the convergence of the integral norm

$$\| u ; L^{q_3}(I, L^m) \|$$

for what it is sufficient $0 < 1/q_3 < \delta(m)$. The restriction among the parameters is given by

$$\frac{n}{s} = 2(\delta(r) + \delta(m)) \qquad (1)$$

$$\frac{2}{q_3} = 1 - \delta(r) \qquad (2)$$

where we have taken $\delta(r) = \delta(r_2)$ (see Eqs at the top of pg. 20 at the beginning of the proof of IVP_∞ in H^1 for HE).

From (1) we obtain

$$2\delta(m) \leq \frac{n}{s} < 4 \qquad (3)$$

since $0 \leq \delta(r), \delta(m) \leq 1$ for $n \geq 3$ and $0 \leq \delta(r), \delta(m) < 1$ for $n = 2$. In the case $n \geq 3$ the equality at the RHS is excluded because we need $q_3 < \infty$. From (2) and the integrability condition $\frac{2}{q_3} = 1 - \delta(r) < 2\delta(m)$ we obtain from (1)

$$2(1 - \delta(m)) < \frac{n}{s} \qquad (4)$$

Comparison of inequalities (3), (4) yields the lower bound

$$1 < \frac{n}{s}$$

So the remaining part of the proof consists in checking that the s_0 of the statement of the proposition allow to solve Eqs. (1) and (2) in the interval

$$\frac{1}{4} \leq \frac{n}{4s} < 1$$

with $r_0 = r$. From the definition of r we have

$$\delta(r) = \begin{cases} \dfrac{n}{4s} & \text{for } \dfrac{1}{2} \leq \dfrac{n}{4s} < 1 \quad \text{Region A} \\[2ex] \dfrac{n}{2s} - \dfrac{1}{2} & \text{for } \dfrac{1}{4} < \dfrac{n}{4s} \leq \dfrac{1}{2} \quad \text{Region B} \end{cases}$$

Region A. Define $\delta(m) = \frac{n}{2s} - \delta(r) = \frac{n}{4s} = \delta(r)$

$$0 < \frac{2}{q_3} = 1 - \delta(r) < 2\delta(r) = 2\,\delta(m) \qquad \text{O.K. integrability}$$

Region B Define $\delta(m) = \frac{n}{2s} - \left(\frac{n}{2s} - \frac{1}{2}\right) = \frac{1}{2}$

$$0 < \frac{2}{q_3} = 1 - \delta(r) < 2\,\delta(m) = 1$$

□

Pseudo-Conformal Law

Def. $M(t) \equiv \exp\left(\frac{i x^2}{2t}\right)$

$$(D_0(t) v)(x) \equiv v(t^{-1} x) \qquad t \neq 0$$

$$D(t) \equiv (it)^{-n/2} D_0(t)$$

$$\begin{cases} (it)^{n/2} \equiv |t|^{n/2} \exp\left(i \frac{n}{2} \frac{\pi}{2} \sigma(t)\right) = |t|^{n/2} \exp\left(i n \frac{\pi}{4} \sigma(t)\right) \\ \sigma(t) = \operatorname{sgn} t \end{cases}$$

Def. For u function of space-time $\mathbb{R}^{n+1}$ define u_c, function of $\mathbb{R}^{n+1}$ by

$$u(t) = M(t) D(t) \overline{u_c(1/t)}. \qquad (*)$$

Explicitly $$u(t,x) = (it)^{-n/2} \exp\left(\frac{i x^2}{2t}\right) \overline{u_c(t^{-1}, t^{-1} x)}$$

Remark. It holds: $u_c(t) = M(t) D(t) \overline{u(1/t)}$

i.e. the transformation $u_c \longrightarrow u$ is an involution.

Proof: By direct check:

$$\overline{D(t)}\, D(1/t) = D(t)\, \overline{D(1/t)} = \mathbb{1}$$

$$M(t)\, D(t) = D(t)\, M(1/t)$$

From $(*)$:
$$\overline{u(1/t)} = \overline{M(1/t)}\, \overline{D(1/t)}\, u_c(t) = \overline{D(1/t)}\, \overline{M(t)}\, u_c(t)$$

$$\therefore \quad u_c(t) = \overline{M(t)}^{-1}\, \overline{D(1/t)}^{-1}\, \overline{u(1/t)}$$

$\square$

The unitary group $U(t)$ can be expressed through $D(\cdot)$, $M(\cdot)$ and F

where $$Fv(\xi) = (2\pi)^{-n/2} \int dx \, e^{-i\xi \cdot x} v(x)$$:

$$U(t) = M(t) \, D(t) \, F \, M(t)$$

In fact

$$(U(t)v)(x) = (2\pi i t)^{-n/2} \int dy \, e^{\frac{i}{2t}(x-y)^2} v(y)$$

$$= e^{(i/2t)x^2} (it)^{-n/2} \int \frac{dy}{(2\pi)^{n/2}} \, e^{-i\frac{x}{t}\cdot y} \, e^{(i/2t)y^2} v(y) .$$

Lemma (1) Let $u \in \mathcal{C}([T,\infty), L^2)$ and $u_+ \in L^2$. Suppose that

$$L^2\text{-}\lim_{t\to\infty} U(t)^{-1} u(t) = u_+$$

Then $u_c \in \mathcal{C}([0, T^{-1}], L^2)$ with $u_c(0) = v = \overline{Fu_+}$

(2) Let $u_c \in \mathcal{C}([0, T^{-1}]; L^2)$ with $u_c(0) = v_+ = \overline{Fu_+}$.

Then $u \in \mathcal{C}([T,\infty), L^2)$ and il existe

$$L^2\text{-}\lim_{t\to\infty} U(t)^{-1} u(t) = u_+ .$$

Proof. From $$u(t) = M(t) \, D(t) \, \overline{u_c(1/t)}$$

and $$U(t) = M(t) \, D(t) \, F \, M(t)$$

we can write

$$\|U(t)^{-1}u(t) - u_+\|_2 = \|u(t) - U(t)u_+\|_2 = \|\overline{u_c(1/t)} - FM(t)u_+\|_2 .$$

Therefore

$$\left| \|U(t)^{-1}u(t) - u_+\|_2 - \|\overline{u_c(1/t)} - Fu_+\|_2 \right|$$

$$= \left| \|\overline{u_c(1/t)} - FM(t)u_+\|_2 - \|\overline{u_c(1/t)} - Fu_+\|_2 \right|$$

$$\leq \|FM(t)u_+ - Fu_+\|_2 = \|M(t)u_+ - u_+\|_2$$

The proof is concluded by the remark that $\lim_{t\to\infty} \|M(t)u_+ - u_+\|_2 = 0$

by the Lebesgue's dominated convergence theorem. $\square$

Useful formulae:

$$\nabla M(t) = M(t)\,(\nabla + i\,t^{-1}x)$$

$$(\nabla + i\,t^{-1}x)\,D(t) = D(t)\,(t^{-1}\nabla + i x)$$

$$\frac{1}{2}\Delta M(t) D(t) = \frac{1}{2} M(t) D(t) (t^{-1}\nabla + i x)^2 = M(t) D(t) \left\{ \frac{1}{2} t^{-2}\Delta - \frac{1}{2}x^2 + i\,t^{-1} x\cdot\nabla + i\,\frac{t^{-1}}{2} n \right\}$$

$$t\partial_t D_0(t) = D_0(t)\,(t\partial_t - x\cdot\nabla)$$

$$i\partial_t D(t) = D(t)\left\{ i\partial_t - i\,t^{-1}\left(x\cdot\nabla + \frac{n}{2}\right)\right\}$$

$$i\partial_t M(t) = M(t)\left(i\partial_t + \frac{t^{-2}}{2}x^2\right)$$

$$i\partial_t M(t) D(t) = M(t) D(t)\left\{ i\partial_t + \frac{1}{2}x^2 - i\,t^{-1}\left(x\cdot\nabla + \frac{n}{2}\right)\right\}$$

$$\therefore \quad \left(i\partial_t + \frac{1}{2}\Delta\right) M(t) D(t) = M(t) D(t) \overline{\left(i\partial_t + \frac{1}{2}t^{-2}\Delta\right)}$$

$$= M(t) D(t)\, t^{-2}\left(i\frac{\partial}{\partial(\frac{1}{t})} + \frac{1}{2}\Delta\right)$$

Finally

$$\left(i\partial_t + \frac{1}{2}\Delta\right) u(t) = \left(i\partial_t + \frac{1}{2}\Delta\right) M(t) D(t)\, \overline{u_c(1/t)} =$$

$$= M(t) D(t)\, t^{-2}\,\overline{\left(i\frac{\partial}{\partial(1/t)} + \frac{1}{2}\Delta\right) u_c\left(\frac{1}{t}\right)} = M(t) D(t) \left\{ \overline{t^2\left(i\frac{\partial}{\partial t} + \frac{1}{2}\Delta\right) u_c(t)} \right\}_{t \mapsto 1/t}$$

We now rewrite in terms of u_c the additional terms

present in the SE.

(LS) Linear SE with potential V depending only on space.

$$V u(t) = V M(t) D(t) u_c(t^{-1}) = M(t) D(t) V(t\,\cdot) u_c(t^{-1})$$

$$\therefore \left(i\partial_t + \tfrac{1}{2}\Delta\right) u(t) - V u(t) = M(t) D(t) t^{-2} \overline{\left\{ \left(i\partial_t + \tfrac{1}{2}\Delta\right) u_c(t) - t^{-2} \overline{V(t^{-1}\cdot)}\, u_c(t) \right\}}_{t \to \frac{1}{t}} \quad (1)$$

$$(NLS): \; |u(t)|^{p-1} u(t) = M(t) D(t) |t|^{-(p-1)n/2} |u_c(t^{-1})|^{p-1} u_c(t^{-1})$$

$$\therefore \left(i\partial_t + \tfrac{1}{2}\Delta\right) u(t) - \lambda |u(t)|^{p-1} u(t) = M(t) D(t) t^{-2} \overline{\left\{ \left(i\partial_t + \tfrac{1}{2}\Delta\right) u_c(t) - \bar\lambda\, t^{\gamma-2} |u_c(t)|^{p-1} u_c(t) \right\}}_{t \to \frac{1}{t}} \quad (2)$$

$$(HE): \left(V * |u(t)|^2\right) u(t) = \left(V * |D(t) u_c(t^{-1})|^2\right) M(t) D(t) \overline{u_c(t^{-1})}$$

$$= M(t) D(t) \left\{ D_0(t)^{-1} \left(V * |D(t) u_c(t^{-1})|^2\right) \right\} \overline{u_c(t^{-1})}$$

Now: $\left(D_0(t)^{-1} \left(V * |D(t) u_c(t^{-1})|^2\right)\right)(x) =$

$$\int dy\, V(tx - y) |u_c(t^{-1}, t^{-1} y)|^2 |t|^{-n} = \int dy\, V(t(x-y)) |u_c(t^{-1}, y)|^2$$

$$\therefore \left(i\partial_t + \tfrac{1}{2}\Delta\right) u(t) - \left(V * |u(t)|^2\right) u(t) = M(t) D(t) t^{-2} \overline{\left\{ \left(i\partial_t + \tfrac{1}{2}\Delta\right) u_c(t) - \overline{V(t^{-1}\cdot)} * |u_c(t)|^2 u_c(t)\, t^{-2} \right\}}_{t \to 1/t} \quad (3)$$

Proposition (short/long range) NLS. Let $1 < p \leq 2$ for $n = 1,2$ and $1 < p \leq 1 + 2/n$ for $n \geq 3$. Let $u_+ \in L^2(\mathbb{R}^n)$ and $u \in \mathcal{C}([T,\infty), L^2)$ for some $T > 0$ be solution of the NLS equation such that

$$\lim_{t\to\infty} \| u(t) - U(t) u_+ \|_2 = 0 .$$

Then $u_+ = 0$. If in addition ($p < 2$ for $n = 2$ and) $\lambda \in \mathbb{R}$, conservation of the L^2-norm yields $u =$

Proof. We recall the equation for u_c

$$i\partial_t u_c + \frac{1}{2}\Delta u_c = \bar{\lambda}\, t^{\gamma-2} |u_c|^{p-1} u_c \qquad (*)$$

where we have set $\gamma = (p-1)\frac{n}{2}$. By the Lemma of PC2

$$u_c \in \mathcal{C}\left([0, T^{-1}], L^2\right).$$

Integration over space with a test function φ $(\in \mathcal{C}_0^\infty(\mathbb{R}^n))$ yields

$$i\partial_t \langle \varphi, u_c \rangle + \frac{1}{2}\langle \Delta\varphi, u_c \rangle = \bar{\lambda}\, t^{\gamma-2} \langle \varphi, |u_c|^{p-1} u_c \rangle .$$

By subsequent integration over time with $0 < t_1 < t_2$ we obtain

$$i\left(\langle \varphi, u_c(t_2)\rangle - \langle \varphi, u_c(t_1)\rangle \right) + \frac{1}{2}\int_{t_1}^{t_2} dt' \langle \Delta\varphi, u_c(t')\rangle =$$

$$\bar{\lambda}\int_{t_1}^{t_2} dt'\, t'^{\gamma-2} \langle \varphi, |v_+|^{p-1} v_+ \rangle + \bar{\lambda}\int_{t_1}^{t_2} dt'\, t'^{\gamma-2}\, \phi(t') \qquad (**)$$

where $\phi(t) = \langle \varphi, (|u_c|^{p-1} u_c - |v_+|^{p-1} v_+) \rangle.$

Obviously to be defined in the equation (*) $|u_c|^p$ must be in some L^k with $1 \leq k \leq \infty$. Now $|u_c|^p \in L^{2/p}$ from the only information we have on u_c, i.e. $u_c \in L^2$.

From the assumption on p $1 \leq k = \frac{2}{p}$, so that, the straightforward estimate

$$\left| |u_c|^{p-1} u_c - |v_+|^{p-1} v_+ \right| \leq C \, \mathrm{Max}\left(|u_c|^{p-1}, |v_+|^{p-1} \right) | u_c - v_+ |$$

implies

$$|\phi(t)| \leq C \|\varphi\|_{2/(2-p)} \mathrm{Max}\left(\|u_c\|_2^{p-1}, \|v_+\|_2^{p-1} \right) \| u_c(t) - v_+ \|_2$$

Since u_c is bounded in bounded intervals and $\lim_{t \to 0} \| u_c(t) - v_+ \|_2 = 0$, we continue the previous estimate in

$$|\phi(t)| \leq C \| u_c(t) - v_+ \|_2 \leq \frac{1}{2} \left| \langle \varphi, |v_+|^{p-1} v_+ \rangle \right|$$

for all $0 \leq t \leq t_0$ with t_0 suitably chosen, provided $\langle \varphi, |v_+|^{p-1} v_+ \rangle \neq 0$. In this case set

$$\langle \varphi, |v_+|^{p-1} v_+ \rangle = |\langle \varphi, |v_+|^{p-1} v_+ \rangle| \, \omega$$

where $|\omega| = 1$. Then $(**)$ implies

$$\left| \bar{\lambda}^{-1} \bar{\omega} \left\{ \left(i\langle \varphi, u_c(t_2)\rangle - i\langle \varphi, u_c(t_1)\rangle \right) + \frac{1}{2} \int_{t_1}^{t_2} dt' \, \langle \Delta\varphi, u_c(t')\rangle \right\} \right|$$

$$\left| \int_{t_1}^{t_2} dt' \, t'^{\gamma-2} |\langle \varphi, |v_+|^{p-1} v_+ \rangle| + \bar{\omega} \int_{t_1}^{t_2} dt' \, t'^{\gamma-2} \phi(t') \right|$$

$$\geq \int_{t_1}^{t_2} dt' \, t'^{\gamma-2} \, |\langle \varphi, |v_+|^{p-1} v_+ \rangle| - \int_{t_1}^{t_2} dt' \, t'^{\gamma-2} |\phi(t')| \geq$$

$$\geq \frac{1}{2} \left(\int_{t_1}^{t_2} dt' \, t'^{\gamma-2} \right) |\langle \varphi, |v_+|^{p-1} v_+ \rangle| \qquad \text{for } t_1, t_2 \leq t_0 . \quad (***)$$

For $\gamma \leq 1$ we can compute

$$-\int_{t_1}^{t_2} dt' \, t'^{\gamma-2} = \begin{cases} (1-\gamma)^{-1} \left(t_1^{\gamma-1} - t_2^{\gamma-1} \right) & \text{for } \gamma < 1 \\ \ln \dfrac{t_2}{t_1} & \text{for } \gamma = 1. \end{cases}$$

The above time integral $\xrightarrow[t_1 \to 0]{} \infty$ so that we reach a contradiction since the LHS of the inequality $(***)$ has a finite limit as $t_1 \to 0$.

Therefore $\forall\ \varphi\ (\in \mathscr{C}_0^\infty(\mathbb{R}^n))\quad \langle \varphi, |v_+|^{p-1} v_+ \rangle = 0$

$\therefore \qquad v_+ = 0 = u_+ .$

- Remark: If p satisfies the assumptions listed in the statement of this proposition (with the minor change of the condition $p \leq 2$ for $n=2$ by $p<2$,) the local Cauchy problem can be solved for $u \in \mathscr{C}(\cdot, L^2)$. To see this look at the IVP for NLS pg. 2 where the two basic equations for the relevant parameters

$$\frac{1}{r_1} + \frac{p}{r} = 1$$

$$\frac{1}{q_1} + \frac{p}{q} = 1 - \theta \qquad\qquad 0 \leq \theta \leq 1$$

can be solved with $r = 2$. This yields

$$(p-1)\frac{n}{2} = \delta(r_1) = \frac{2}{q_1} = 2(1-\theta)$$

with the conditions of admissibility for (q_1, r_1), namely

$$\delta(r_1) \leq 1/2 \qquad \text{for } n = 1$$
$$< 1 \qquad \text{for } n = 2$$
$$\leq 1 \qquad \text{for } n \geq 3$$

which gives $1 < p \leq 2$ for $n=1$, $1 < p < 2$ for $n=2$ and $1 < p \leq 1 + \frac{2}{n}$ for $n \geq 3$. ✓

- By uniqueness in $\mathscr{C}(\cdot, L^2)$ it follows that $u \in L^q_{loc}(\cdot, L^2)$ for all (q, r) admissible. With that regularity, for $\lambda \in \mathbb{R}$, one can prove the conservation of the L^2-norm, i.e. $\|u(t)\|_2 = \|u_+\|_2$

Finally we conclude $\quad u(t) = 0 .$

$\square$

<u>Proposition (short/long range) HE.</u> Let $0<\gamma<1/2$ for $n=1$, $0<\gamma<1$ for $n=2$ and $0<\gamma\leq 1$ for $n\geq 3$. Let $u_+\in L^2$ and $u\in \mathcal{C}([T,\infty), L^2)$ for some $T>0$ be solution of the HE with $V(x)=\lambda|x|^{-\gamma}$ such that

$$\lim_{t\to\infty}\|u(t)-U(t)u_+\|_2=0.$$

Then $u_+=0$. Furthermore if $\gamma<1$ for $n\geq 3$ and if $\lambda\in\mathbb{R}$, conservation of the L^2-norm yields $u=0$.

<u>Proof</u> The proof is exactly the same as for NLS, with the equation (*) replaced by the corresponding one for u_c coming from the HE. In this situation the (**) equation is replaced by the same LHS while the RHS becomes

$$\bar\lambda\int_{t_1}^{t_2}dt'\,t'^{\gamma-2}\langle\varphi,(|x|^{-\gamma}*|v_+|^2)v_+\rangle+\bar\lambda\int_{t_1}^{t_2}dt'\,t'^{\gamma-2}\phi(t')$$

where now $$\phi(t)=\left\langle\varphi,\left((|x|^{-\gamma}*|u_c(t)|^2)u_c(t)-(|x|^{-\gamma}*|v_+|^2)v_+\right)\right\rangle$$

We now write $$V_1=\begin{cases}|x|^{-\gamma} & \text{for } |x|\geq 1\\ 0 & \text{for } |x|<1\end{cases}$$

$$V_2=\begin{cases}0 & \text{for } |x|<1\\ |x|^{-\gamma} & \text{for } |x|\geq 1\end{cases}$$

so that $$|x|^{-\gamma}=V_1(x)+V_2(x)$$

Notice that $V_1\in L^{s_1}$, $V_2\in L^{s_2}$ with $s_1=\frac{n}{(\gamma-\varepsilon)}$, $s_2=\frac{n}{\gamma+\varepsilon}$.

Application of the Young's inequality implies

$$|\phi(t)| \leq \sum_{j=1}^{2} \|\varphi\|_{z_j} \|V\|_{s_j} \, \mathrm{Max}\left(\|u_c(t)\|_2^2, \|v_+\|_2^2\right) \|u_c(t) - v_+\|_2$$

$$\frac{n}{s_j} = \delta(r_j) \iff \frac{1}{s_j} + \frac{1}{z_j} = \frac{1}{2}$$

which implies $z_j \geq 2$, $s_j \geq 2$. Now $s_j \geq 2$ is equivalent

to $\frac{\gamma \pm \varepsilon}{n} \leq \frac{n}{2}$ which is satisfied by our assumptions.

By exactly the same arguments as for NLS we obtain

$$(|x|^{-\gamma} * |v_+|^2) v_+ = 0$$

This means that $(|\cdot|^{-\gamma} * |v_+|^2)(x) \, v_+(x) = 0$ almost everywhere

i.e $\exists \, N \subset \mathbb{R}^n$ of measure zero such that $\forall \, x \in \mathbb{R}^n \setminus N \equiv A$

$$(|\cdot|^{-\gamma} * |v_+|^2)(x) \; v_+(x) = 0 \qquad \boxtimes$$

If $(|\cdot|^{-\gamma} * |v_+|^2)(x) \neq 0 \quad \forall \, x \in A$ equality $\boxtimes$ implies

$$v_+(x) = 0 \quad \forall \, x \in A$$

If $\exists \, x_0 \in A$ such that $(|\cdot|^{-\gamma} * |v_+|^2)(x_0) = 0$ $\exists \, M \subset \mathbb{R}^n$

of measure zero such that $\forall \, y \in \mathbb{R}^n \setminus M \equiv B$

$$|x_0 - y|^{-\gamma} |v_+(y)|^2 = 0$$

This implies that $|v_+(y)| = 0 \quad \forall \, y \in B \setminus \{x_0\}$, and therefore

$$v_+ = 0 = u_+$$

- Remark: If γ satisfies the assumptions listed in the statement of this proposition (with the minor change of the condition $\gamma \leq 1$ for $n \geq 3$ by $\gamma < 1$) the local Cauchy problem for HE with $V = V_1 + V_2$,

$V_j \in L^{s_j}$, $s_j = n/(\gamma \pm \varepsilon)$, $j = 1,2$, can be solved with $u \in \mathcal{C}(\cdot, L^2)$.

To prove this look at the IVP for HE pag. 4 where the two basic equations for the relevant parameters

$$1/r_1 + 3/r + 1/s = 2$$

$$1/q_1 + \gamma/q = 1 - \theta \qquad 0 \leq \theta \leq 1$$

can be solved with $r = 2$ (Here we treat separately the potentials V_1 and V_2). This yields

$$\frac{n}{s} = \delta(r_1) = \frac{2}{q_1} = 2(1-\theta).$$

The usual restriction on r_1 provide exactly the conditions on γ: $0 < \gamma < 1/2$ for $n = 1$ and $0 < \gamma < 1$ for $n \geq 2$.

- By uniqueness in $\mathcal{C}(\cdot, L^2)$ it follows that $u \in L^q_{loc}(\cdot, L^2)$ for all (q,r) admissible. With that regularity, for $\lambda \in \mathbb{R}$, one can prove the conservation of the L^2-norm, i.e. $\|u(t)\|_2 = \|u_+\|_2$.

Finally we conclude $u(t) = 0$

□

Proposition (short/long range) LS. Let V be a real function PC13

where $V(x) = \begin{cases} V_1(x) & \text{for } |x| \leq a \\ \lambda |x|^{-\gamma} & \text{for } |x| \geq a \end{cases}$ for some $a > 0$,

with V_1 real function, $\operatorname{Supp} V_1 \subset \{x : |x| \leq a\}$, $V \in L^r(\mathbb{R}^n)$ with

$r \geq \operatorname{Max}(2, \frac{n}{2})$ for $n \neq 4$ and $r > 2$ for $n = 4$, $\lambda \in \mathbb{R}$, $0 < \gamma \leq 1$.

Let $u_+ \in L^2$ and let $u(t)$ be solution of the LS (The

LS equation can be solved globally in time by the

unitary group $W(t)$, with generator $H = -\frac{1}{2}\Delta + V$, which is a

Kato perturbation of $-\frac{1}{2}\Delta$. If u is solution of the

LS $\quad i\partial_t u + \frac{1}{2}\Delta u = Vu$ in the weakest sense then

$\forall \varphi \in \mathcal{D}(H)$

$$i\partial_t \langle W(t)\varphi, u(t)\rangle = \langle(-\tfrac{1}{2}\Delta + V)W\varphi, u(t)\rangle$$

$$+ \langle W(t), (-\tfrac{1}{2}\Delta + V)u(t)\rangle = 0$$

$$\therefore \quad \langle \varphi, W(t)^{-1} u(t)\rangle = \langle \varphi, u(0)\rangle$$

$$\therefore \quad u(t) = W(t)\, u(0) \quad)$$

such that

$$\lim_{t \to \infty} \| u(t) - U(t) u_+ \|_2 = 0 .$$

Then $u(t) = 0 \ \forall t$.

Proof. The proof is similar to that for NLS and HE.

The equation for u_c is

$$i\partial_t u_c + \frac{1}{2}\Delta u_c = t^{-2}\left(V_1\left(\frac{\cdot}{t}\right)\chi_{|x|\le at} + \lambda\, t^{-\gamma}\, |\cdot|^{-\gamma}\chi_{|x|\ge at}\right)u_c$$

Integration over space with a test function $\varphi \in \mathcal{C}_0^\infty(\mathbb{R}^n)$ and

$\mathrm{Supp}\,\varphi \subset \{x : 0 < r \le |x| \le R\}$ yields for $t \le t_0 = \frac{r}{2a}$

$$i\partial_t \langle \varphi, u_c\rangle + \frac{1}{2}\langle \Delta\varphi, u_c\rangle = \lambda\, t^{\gamma-2}\langle \varphi, |\cdot|^{-\gamma} u_c\rangle$$

where the term with V_1 vanishes since $(\mathrm{Supp}\,\varphi)\cap \mathrm{Supp}\,\chi_{|x|\le at}) = \phi$

By integration over the interval $[t_1, t_2]$ with $t_1 < t_2 \le t_0$

we obtain

$$i\Big(\langle\varphi, u_c(t_2)\rangle - \langle \varphi, u_c(t_1)\rangle\Big) + \frac{1}{2}\int_{t_1}^{t_2} dt' \,\langle \Delta\varphi, u_c(t')\rangle =$$

$$= \lambda \int_{t_1}^{t_2} dt'\, t'^{\gamma-2}\langle\varphi, |\cdot|^{-\gamma}\chi_{|x|\ge at'}v_+\rangle + \lambda\int_{t_1}^{t_2} dt'\, t'^{\gamma-2}\phi(t')$$

with $\phi(t) = \Big\langle \varphi, \Big(|\cdot|^{-\gamma}\chi_{|x|\ge at}\big(u_c(t) - v_+\big)\Big)\Big\rangle$

By the condition $t_1, t_2 \le t_0$,

$$\langle \varphi, |\cdot|^{-\gamma} \chi_{|x|\geq at}\, v_+ \rangle = \langle \varphi, |\cdot|^{-\gamma} v_+ \rangle = \int_{r\leq |x|\leq R} dx\, \overline{\varphi(x)}\, |x|^{-\gamma} v_+(x)$$

and

$$\langle \varphi, |\cdot|^{-\gamma} \chi_{|x|\geq at} (u_c(t) - v_+) \rangle = \langle \varphi, |\cdot|^{-\gamma} (u_c(t) - v_+) \rangle$$

$$= \int_r^R dx\, \overline{\varphi(x)}\, |\cdot|^{-\gamma} (u_c(t,x) - v_+(x))$$

From now on the argument proceeds as for NLS and HE. This implies $v_+ = 0$ (The fact that we use as test function space $C_0^\infty(\mathbb{R}^n \setminus \{0\})$ instead than $C_0^\infty(\mathbb{R}^n)$ is irrelevant).

In this linear case we have conservation of L^2-norm so that

$$\|u(t)\|_2 = \| \cdot \|_2 = \| v_+ \|_2 = 0$$

$\square$

Asymptotic completeness for NLS in $Y^1(\mathbb{R}_+)$.

Proposition (AC in $Y^1(\mathbb{R})$). Let $\frac{4}{n} \leq p-1$ and in addition $p-1 < 4/(n-2)$ for $n \geq 3$. Let $\lambda > 0$. Let $u \in Y^1_{loc}(\mathbb{R}_+)$ be solution of NLS. Then $u \in Y^1(\mathbb{R}_+)$.

Proof. We will use some a priori estimates to extract some time decay. Recall PC inversion

$$u(t) = M(t) D(t) \overline{u_c(1/t)}$$

$$\therefore \quad J(t) u(t) = J(t) M(t) D(t) \overline{u_c(1/t)} = M(t) D(t) \overline{(-i\nabla u_c(1/t))}$$

$$J(t) u_c(t) = M(t) D(t) \overline{(-i\nabla u(1/t))}$$

so that

$$\begin{pmatrix} u(t) \\ \nabla u(t) \\ J(t) u(t) \end{pmatrix} \in L^q(\{a, b\}; L^2) \qquad 0 < a < b < \infty$$

is equivalent to

$$\begin{pmatrix} u_c(t) \\ \nabla u_c(t) \\ J(t) u_c(t) \end{pmatrix} \in L^q(\{b^{-1}, a^{-1}\}; L^2)$$

puisque $\quad d\frac{1}{t} = \frac{1}{t^2} dt$.

Similarly $\begin{pmatrix} u(\cdot) \\ \nabla u(\cdot) \\ J(\cdot) u(\cdot) \end{pmatrix} \in \mathcal{C}((0,\infty); L^2) \leftrightarrow \begin{pmatrix} u_c(\cdot) \\ \nabla u_c(\cdot) \\ J(\cdot) u_c(\cdot) \end{pmatrix} \in \mathcal{C}((0,\infty); L^2)$.

The "conservation of the energy" for u_c is obtained from the equation for u_c:

$$\partial_t \Big\{ \underbrace{\tfrac{1}{2}\|\nabla u_c(t)\|_2^2 + t^{\gamma-2}\frac{2}{(p+1)}\int dx\, \lambda\, |u_c(t,x)|^{p+1}}_{m(t)} \Big\} = (\gamma-2)\, t^{\gamma-3}\frac{2}{(p+1)}\int dx\, \lambda\, |u_c(t,x)|^{p+1}$$

where $\gamma = (p-1)\frac{n}{2}$. This can be proved formally. The effective proof requires a regularization to be removed at the end.

If $\gamma \geq 2$ $\quad \partial_t m(t) \geq 0$ so that $\forall\ 0 < t \leq 1$

$$m(t) \leq m(1)$$

$$\therefore \quad \|J(t)u(t)\|_2 \leq M(u(1)) \quad \forall \quad 1 \leq t < \infty .$$

This estimate implies:

$$\|u(t)\|_r \leq C\, \|u(t)\|_2^{1-\delta(r)}\, \|J(t)u(t)\|_2^{\delta(r)}\, |t|^{-\delta(r)} \leq M(u(1))\, t^{-\delta(r)}$$

with $0 \leq \delta(r) \leq 1/2$ for $n=1$, $0 \leq \delta(r) < 1$ for $n=2$, $0 \leq \delta(r) \leq 1$ for $n \geq 3$

- Now use the IE to exhibit the decay of all the relevant quantities which are $u(\cdot)$, $\nabla u(\cdot)$, $J(\cdot)u(\cdot) \in L^q([1,\infty), L^r)$ for (q,r) admissible. Rewrite:

$$u(t) = U(t-t_0)\,u(t_0) - i\int_{t_0}^{t} dt'\, U(t-t')\, \lambda\, |u(t')|^{p-1} u(t')$$

and in general

$$(Au)(t) = U(t-t_0)(Au)(t_0) - i\int_{t_0}^{t} dt'\, U(t-t')\, A\left(|u(t')|^{p-1}u(t')\right)$$

where $A = ($ identity, ∇, J $)$ and we have used

$$J(t)\, U(t-t') = U(t-t')\, J(t').$$

By the same estimates as in the solution of the Cauchy problem at ∞ in H^1 we can write by Strichartz:

$$\| Au ; L^q([t_0, t_1]; L^r \| \leq c_0 \|(Au)(t_0)\|_2 +$$

$$\| Au ; L^q([t_0, t_1]; L^r\| \cdot c_1 \| u ; L^{q_3}([t_0, t_1]; L^m) \|^{p-1}.$$

with

$$\begin{cases} \dfrac{2}{r} + \dfrac{p-1}{m} = 1 \\[2ex] \dfrac{2}{q} + \dfrac{p-1}{q_3} = 1 \end{cases}$$

$$(p-1)\left(\frac{n}{2} - \delta(m)\right) = 2\,\delta(r)$$

$$(p-1)\left(\frac{n}{2} - \delta(m) + \frac{2}{q_3}\right) = 2$$

the last one from linear combination of the two initial equalities

The previous estimate is valid $\forall\, 0 \leq t_0, t_1 < \infty$. We want to show that with suitable choice of r q_3 m

$u \in L^{q_3}([0,\infty); L^m)$ i.e. $\int_0^\infty dt\, \|u(t)\|_m^{q_3} < \infty$.

Choose: $\delta(p+1) = \delta(m) = \delta(r)$ which satisfies $(p-1)\left(\frac{n}{2} - \delta(p+1)\right) = 2\delta(p+1)$

$\therefore \quad (p-1)\left(\frac{n}{2} - \delta(p+1) + \frac{2}{q_3}\right) = 2$

$\therefore \quad \delta(p+1) + \frac{p-1}{q_3} = 1 \qquad \boxtimes$

For $n=1$ $\delta(p+1) \leq 1/2$, for $n=2$ $\delta(p+1) < 1$, for $n \geq 3$ $\delta(p+1) \leq 1$

(puisque $p-1 < \frac{4}{n-2}$)

$\therefore \quad 0 < q_3 < \infty$.

De $\boxtimes$ on a que $q_3\, \delta(p+1) > 1$ (qui implique $q_3 > 1$)

$\iff \quad \delta(p+1) + (p-1)\,\delta(p+1) > 1 \iff p\,\delta(p+1) > 1$

La fonction $p \longrightarrow p\,\delta(p+1)$ est croissante.

Il suffit de contrôler que $p\,\delta(p+1) > 1$

pour $p = \bar{p} = 1 + \frac{4}{n}$.

$$p\,\delta(p+1) = \left(1+\frac{4}{n}\right)\left(\frac{n}{2} - \frac{n}{2}\,\frac{1}{1+\frac{2}{n}}\right) = \left(2+\frac{n}{2}\right)\frac{2/n}{1+2/n}$$

$$= \frac{n+4}{n+2} > 1$$

The lower bound on p could be lowered.

With that choice for r, q_3, m we take t_0 such that

$$c_1 \left\| u;\, L^{q_3}([t_0,\infty), L^m)\right\|^{p-1} \leq 1/2$$

so that

$$\| Au;\, L^q([t_0,t_1], L^r)\| \leq 2c_0 \|Au(t_0)\|_2$$

$\forall \quad t_0 < t_1$

$$\therefore \quad \| Au;\, L^q([t_0,\infty), L^r)\| < \infty$$

et donc $u \in Y^1(\mathbb{R}^+)$. $\square$