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Renormalization at any order of lattice infrared QED_4

(Joint work in progress with A. Giuliani & V. Mastropietro)

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Introduction

Constructive QFT

- *Aim of CQFT*: Define interacting QFT by constructing euclidean correlations satisfying a suitable set of axioms.
- *Techniques of CQFT*: Rigorous path integral formulation (finite volume, UV and IR cutoffs) + Wilsonian RG; Invariant measures of Stochastic PDEs + Stochastic Quantization; Perturbative and non-perturbative AQFT.
- *Successful Applications*:
  - UV φ^4_d, $d = 2, 3$ (triviality conjecture in $d = 4$)
  - IR φ^4_4 in $d \geq 4$, and in $d \leq 3$ with long range interactions
  - UV Thirring and IR Luttinger models in $d = 2$
  - IR graphene ($d = 2 + 1$) and Weyl semimetal ($d = 3 + 1$)
  - QED_4 with massive photon
  - IR QED_d, $d \geq 2$ and large electron mass
  - IR non-relativistic QED
  - UV QED_3
  - UV $SU(N)$ Yang-Mills in $d = 3, 4$
- *Main Contributors*: Aizenman, Ammari, Balaban, Benfatto, Brydges, Buchholz, Chatterjee, Derezinski, Dimock, Duminil-Copin, Feldman, Fredenhagen, Fröhlich, Gallavotti, Gawedzki, Gérard, Giuliani, Glimm, Gubinelli, Guerra, Guth, Haag, Hairer, Hiroshima, Hofmanova, Hurd, Jaffe, Kupiainen, Lieb, Magnen, Mastropietro, Mattis, Nelson, Pizzo, Porta, Rivasseau, Rosen, Rychkov, Scoppola, Segal, Seiler, Seneor, Simon, Spencer, Spohn, Velo, Wightman, ...

QED_4?

- The list above does not include: QED_4 with massless photons and small electron mass, and the electroweak theory.
- Both theories are expected to be asymptotically free in the IR, and to have a Landau pole in the UV.
- ⚠ Small hope to remove the UV cutoff ⚠
- *Modern point of view*: QED, EW are effective theories, valid only below the great unification scale.
- No obstacle to the IR construction, even non-perturbative.

Perturbative IR QED_4

- At present, there is not even a satisfactory construction of infrared QED_4 at all orders in renormalized perturbation theory!

- It is crucial to reconcile Gauge Invariance/Ward Identities with the Wilsonian RG.

- There are two important works in the literature on the perturbative renormalizability of IR QED_4 that are *intrinsically perturbative*: Feldman-Hurd-Rosen-Wright (1988) and Keller-Kopper (1996).

FHRW. They use Gallavotti-Niccolò trees. In order to preserve the Ward identities, they introduce a *loop regularization* via a fermionic and two bosonic auxiliary fields, with action

$$\sum_{i=1}^{3} \bar{\phi}_i(-i\partial\!\!\!/ + M_i + eA\!\!\!/)\phi_i \ ;$$

these additional terms make the model unstable.

KK. They use Polchinski's flow equations. Non-gauge invariant counterterms such as A^2 and A^4 are added, that could make the model unstable if they have the wrong sign (for the A^4 term, see Bonini-D'Attanasio-Marchesini (1994)).

- Another interesting work, Dimock-Hurd (1992), assumes a *very large electron mass* (10^2 ratio with UV cutoff scale compared to 10^{-10} physical ratio):

DH. They integrate out the electron field, and use Brydges-Yau techniques for the effective photon theory, taking crucial advantage of the large electron mass.

Outline of main results

- Our results concern perturbative IR QED_4 at all orders, using a construction that overcome the previous drawbacks, and hopefully paves the way to a non-perturbative treatment. [We extend ideas previously developed for Weyl semimetals (IR QED_4 with massive photon) by Giuliani-Mastropietro-Portal (2012)]
- Massless electron (small electron mass as a corollary).
- UV cutoff (not to be removed) introduced in a "gauge-invariant" way by setting the model on a rectangular lattice, with lattice spacing ℓ.
- One single gauge-invariant counterterm, for the electron mass.
- Regularized model non-perturbatively well-defined, for small enough bare electron mass counterterm, and electric charge.
- The electric charge flows to zero in the IR (asymptotic freedom), driving the photon wavefunction renormalization to diverge logarithmically as expected (and canceling the spurious non-gauge-invariant terms generated by the RG flow, see below).

Lattice QED_4

The Bare Action

$$S_0(\psi, A) = \frac{\ell^4}{2} \sum_{x\in\Lambda} \Big(\sum_{\mu<\nu} |\mathrm{d}A(x,\mu,\nu)|^2 + \sum_{\mu} [\bar\psi(x)\gamma_\mu \mathrm{d}_A\psi(x,\mu) - \mathrm{d}_A\bar\psi(x,\mu)\gamma_\mu\psi(x) + r\ell\,\mathrm{d}_A\bar\psi(x,\mu)\mathrm{d}_A\psi(x,\mu)] + 2\ell^{-1}\nu_N\bar\psi(x)\psi(x) \Big)$$

- $\Lambda = \ell\mathbb{Z}^4/L\mathbb{Z}^4$
- $\ell = \ell_0 2^N$ lattice spacing, $L(\to\infty)$ size of the box
- $x \in \Lambda$ lattice vertices, $(x,\mu) \in \Lambda^1$ oriented edges, $(x,\mu,\nu) \in \Lambda^2$ oriented faces $(\mu < \nu)$, ...
- $\{\psi(x), \bar\psi(x)\}_{x\in\Lambda}$ Electron field (Grassmann field with 4 spinor components, antiperiodic b.c.)
- $\{A(x,\mu)\}_{(x,\mu)\in\Lambda^1}$ Photon field (real 1-form, with periodic b.c.)

- r is the Wilson mass (to avoid the appearance of "unphysical particles" in the lattice model – preferred by us to fermion doubling)
- ν_N is the electron mass counterterm
- $\mathrm{d}A(x,\mu,\nu) = \ell^{-1}\Big(A(x,\mu) - A(x,\nu) - A(x+\ell_\mu,\mu) + A(x+\ell_\mu,\nu)\Big)$
- $\mathrm{d}_A\psi(x,\mu) = \ell^{-1}\Big(e^{i\ell e_N A(x,\mu)}\psi(x+\ell_\mu) - \psi(x)\Big)$
- $\mathrm{d}_A\bar{\psi}(x,\mu) = \ell^{-1}\Big(e^{-i\ell e_N A(x,\mu)}\bar{\psi}(x+\ell_\mu) - \bar{\psi}(x)\Big)$
- e_N is the bare electric charge

Remark

$S_0\Big(\sqrt{Z_N^\psi}\psi, \sqrt{Z_N^A}A\Big)$ is invariant under the local $U(1)$ gauge transformation

$$\begin{cases} \bar{\psi}(x) \longmapsto \bar{\psi}(x)e^{ie_N\sqrt{Z_N^A}\alpha(x)} \\ \psi(x) \longmapsto \psi(x)e^{-ie_N\sqrt{Z_N^A}\alpha(x)} \\ A(x,\mu) \longmapsto A(x,\mu) + \mathrm{d}_0\alpha(x,\mu) \end{cases}$$

Gauge Fixing

- The average of observables

$$\langle O\rangle_\Lambda = \frac{\int \mathscr{D}\psi \int \mathscr{D}A e^{-S_0\left(\sqrt{Z_N^\psi}\psi,\sqrt{Z_N^A}A\right)} O(\psi,A)}{\int \mathscr{D}\psi \int \mathscr{D}A e^{-S_0\left(\sqrt{Z_N^\psi}\psi,\sqrt{Z_N^A}A\right)}}$$

does not make sense due to the existence of null directions yielded by gauge transformations, making the integrals divergent.

- Gauge fixing:

$$S_\xi(\psi,A) = \frac{\ell^4}{2}\sum_{x\in\Lambda}\Bigg(\sum_{\mu<\nu}|\mathrm{d}A(x,\mu,\nu)|^2 + \xi|\mathrm{d}^*A(x)|^2 + \sum_\mu[\bar\psi(x)\gamma_\mu \mathrm{d}_A\psi(x,\mu) - \mathrm{d}_A\bar\psi(x,\mu)\gamma_\mu\psi(x) + r\ell \mathrm{d}_A\bar\psi(x,\mu)\mathrm{d}_A\psi(x,\mu)] + 2\ell^{-1}\nu_N\bar\psi(x)\psi(x)\Bigg)$$

with $\xi > 0$, $\mathrm{d}^*A(x) = \ell^{-1}\sum_\mu(A(x-\ell_\mu,\mu) - A(x,\mu))$.

Breaking gauge invariance (part I)

- $\boxed{\langle O\rangle_{\xi;\Lambda}\text{ is still ill-defined}}$: there are still null directions due to constant $A(\cdot,\mu)$ (zero-modes of A).

- We introduce an infrared cutoff h^* in the photon covariance:

$$\boxed{\langle O\rangle_{\xi;h^*,\Lambda} = \frac{\int P(\mathscr{D}\psi)\int P_{\xi;\geq h^*}(\mathscr{D}A)e^{-V^{(N)}\left(\sqrt{Z_N^\psi}\psi,\sqrt{Z_N^A}A\right)}O(\psi,A)}{\int P(\mathscr{D}\psi)\int P_{\xi;\geq h^*}(\mathscr{D}A)e^{-V^{(N)}\left(\sqrt{Z_N^\psi}\psi,\sqrt{Z_N^A}A\right)}}}$$

- $P(\mathscr{D}\psi)$ is the Grassmann Gaussian measure with covariance

$$g(x-y) = \frac{1}{Z_N^\psi}\int đk\,\frac{e^{-ik\cdot(x-y)}}{-i\not{s}(k)+M_N(k)}$$

with $\not{s}(k) = \ell^{-1}\sum_\mu \gamma_\mu \sin(\ell k_\mu)$ and $M_N(k) = 2r\ell^{-1}\sum_\mu \sin^2(\frac{1}{2}\ell k_\mu)$.

- $P_{\xi;\geq h^*}(\mathscr{D}A)$ is the Gaussian measure with covariance

$$G_{\xi;\mu\nu}(x-y) = \int \text{đ}k\, e^{-ik\cdot(x-y)} \frac{1-\chi_{h^*}(k)}{|\sigma(k)|^2}\Big(\tfrac{1}{Z_N^A}\Big(\delta_{\mu\nu} - \frac{\sigma_\mu(k)\overline{\sigma_\nu(k)}}{|\sigma(k)|^2}\Big) + \frac{1}{\xi}\frac{\sigma_\mu(k)\overline{\sigma_\nu(k)}}{|\sigma(k)|^2}\Big)\,.$$

where $\sigma(k) = \big(\sigma_0(k), \sigma_1(k), \sigma_2(k), \sigma_3(k)\big)$, $\sigma_\mu(k) = \frac{i}{\ell}(e^{-i\ell k_\mu} - 1)$, and $\chi_{h^*}(k) = \chi(2^{-h^*}\ell_0 k)$ with χ smooth, radial, compactly supported, monotone decreasing in $|\cdot|$, such that $\chi(k) = 1$ for $|k| \leq 1$ and $\chi(k) = 0$ for $|k| \geq 2$.

- $V^{(N)}(\psi, A) = S_0(\psi, A) - \frac{\ell^4}{2}\sum_{x\in\Lambda}\Big(\sum_{\mu<\nu}|\mathrm{d}A(x,\mu,\nu)|^2 +$
$\sum_\mu(\bar\psi(x)\gamma_\mu \mathrm{d}_0\psi(x,\mu) - \mathrm{d}_0\bar\psi(x)\gamma_\mu\psi(x) + r\ell\mathrm{d}_0\bar\psi(x,\mu)\mathrm{d}\psi(x,\mu))\Big)$.

Proposition

$\langle O \rangle_{\xi;h^*,\Lambda}$ *is well-defined (the functional integral is analytic) for* e_N, ν_N *small.*

Remark

- ⚠ In addition to gauge fixing, we broke gauge invariance with h^* ⚠
- This breaking of gauge invariance is "soft enough" to preserve the crucial Ward Identities: V^N is still gauge invariant.

Lemma

If $O(\psi, A)$ is gauge-invariant, then $\langle O\rangle_{\xi;h^*,\Lambda} \equiv \langle O\rangle_{h^*,\Lambda}$ is independent of ξ.

- Hence we choose $\xi \to \infty$ (Lorenz gauge), making the photon propagator purely transverse

$$G_{\mu\nu}(x-y) = \frac{1}{Z_N^A}\int đk\, e^{-ik\cdot(x-y)}\frac{1-\chi_{h^*}(k)}{|\sigma(k)|^2}\Big(\delta_{\mu\nu} - \frac{\sigma_\mu(k)\overline{\sigma_\nu(k)}}{|\sigma(k)|^2}\Big)\,.$$

- We can remove both cutoffs $L \to \infty$ (before, for convenience) and $h^* \to -\infty$, hence we define the averages (we are only able to show their existence perturbatively – *i.e.* for a truncated model – see below)

$$\boxed{\langle O\rangle := \lim_{h^*\to-\infty}\lim_{L\to\infty}\langle O\rangle_{\infty;h^*,\Lambda}}$$

- To study correlations, we also introduce the generating function

$$e^{W_{h^*,\Lambda}(\phi,J)} = \int P(\mathscr{D}\psi)\int P_{\infty;\geq h^*}(\mathscr{D}A)e^{-V^{(N)}\left(\sqrt{Z_N^\psi}\psi,\sqrt{Z_N^A}(A+J)\right)+Z_N^\psi(\phi,\psi)}\,.$$

Ward Identities

- Gauge invariance of the interaction (and the properties of Gaussian integration) imply that

$$W_{h^*,\Lambda}(\phi, J) = W_{h^*,\Lambda}(e^{-ie_N\sqrt{Z_N^A}\alpha}\phi, J + \mathrm{d}\alpha)\ .$$

- Differentiating the above w.r.t. α we get exact Ward Identities on the lattice:

$$\boxed{\sum_\mu \sigma_\mu(p)\hat{\Pi}_{\mu,\mu_1,\ldots,\mu_{n-1}}(p, p_1, \ldots, p_{n-2}) = 0}$$

$$\boxed{\sum_\mu \sigma_\mu(p)\hat{\Gamma}_\mu(p,k) = e_N\sqrt{Z_N^A}\Big(\hat{\mathfrak{g}}(k) - \hat{\mathfrak{g}}(k+p)\Big)}$$

- $$\Pi_{\mu_1,\dots,\mu_n}(x_1,\dots,x_n) = \frac{\delta^n W_{h^*,\Lambda}(0,J)}{\delta J(x_1,\mu_1)\cdots\delta J(x_n,\mu_n)}\Big|_{J=0}$$
- $$\Gamma_\mu(x,y,z) = \frac{\delta^3 W_{h^*,\Lambda}(\phi,J)}{\delta J(x,\mu)\delta\phi(z)\delta\bar{\phi}(y)}\Big|_{\phi=J=0}$$
- $$\mathfrak{g}(x,y) = \frac{\delta^2 W_{h^*,\Lambda}(\phi,0)}{\delta\phi(y)\delta\bar{\phi}(x)}\Big|_{\phi=0}$$

Renormalization Group Analysis

Multiscale slicing of the integral

- $g = g^{(\leq N-1)} + g^{(N)}$, $G_{\mu\nu} = G_{\mu\nu}^{\leq N-1} + G_{\mu\nu}^{(N)}$
- $g^{(N)}, G_{\mu\nu}^{(N)}$ supported on $k \sim \ell_0^{-1} 2^N$ (using χ_N).
- $\psi = \psi^{(\leq N-1)} + \psi^{(N)}$, $A = A^{\leq N-1} + A^{(N)}$
- $P(\mathscr{D}\psi) = P_{\leq N-1}(\mathscr{D}\psi^{(\leq N-1)}) \times P_N(\mathscr{D}\psi^{(N)})$,
  $P_{\infty;\geq h^*}(\mathscr{D}A) = P_{[h^*,N-1]}(\mathscr{D}A^{\leq N-1}) \times P_N(\mathscr{D}A^{(N)})$
- Now we can integrate $P_N(\mathscr{D}\psi^{(N)})$ and $P_N(\mathscr{D}A^{(N)})$ out, and iterate by then integrating out slices of momenta $\sim 2^{N-1}, \ldots, 2^h, \ldots$

Generating function at scale $h > h^*$

$$e^{W_{h^*,\Lambda}(\phi,J)} = e^{|\Lambda| \sum_{h<k\leq N} E^{(k)}} \int P_{\leq h}(\mathscr{D}\psi) \int P_{[h^*,h]}(\mathscr{D}A) e^{-V^{(h)}\left(\sqrt{Z_h^\psi}\psi, \sqrt{Z_h^A}(A+J), \sqrt{Z_h^\psi}\phi\right)}$$

- The fermionic measure is Gaussian with covariance

$$\hat{g}^{(\leq h)}(k) = \frac{1}{Z_h^\psi} \frac{\chi_h(k)}{-i\not{s}(k) + M_h(k)}\ ,\ M_h(k) = \frac{Z_N^\psi}{Z_h^\psi} M_N(k)\ .$$

- The bosonic measure is Gaussian with covariance

$$\hat{G}_{\mu\nu}^{(\leq h)}(k) = \frac{1}{Z_h^A} \frac{\chi_h(k) - \chi_{h^*}(k)}{|\sigma(k)|^2} \left(\delta_{\mu\nu} - \frac{\sigma_\mu(k)\overline{\sigma_\nu(k)}}{|\sigma(k)|^2}\right)\ .$$

- $\boxed{V^{(h)} = \mathscr{L}V^{(h)} + \mathscr{R}V^{(h)}}$
- $\mathscr{R}V^{(h)} = \mathscr{R}V_{\mathrm{B}}^{(h),\infty} + \mathscr{R}V_{\mathrm{FSE}}^{(h)}$
- Using discrete symmetries,

$$\mathscr{L}V^{(h)}(\psi, A, \phi) = 2^h\tilde{\nu}_h\sum_x\bar{\psi}(x)\psi(x) + \frac{\tilde{z}_h^\psi}{2}\sum_{x,\mu}(\bar{\psi}(x)\gamma_\mu \mathrm{d}\psi(x,\mu) - \mathrm{d}\bar{\psi}(x,\mu)\gamma_\mu\psi(x))$$
$$+ \tilde{\eta}_h(\phi,\psi) + i\tilde{e}_h\sum_x\bar{\psi}(x)\not{A}(x)\psi(x) + 2^{2h}\tilde{m}_h\sum_{x,\mu}A(x,\mu)^2 + \frac{\tilde{z}_h^A}{2}\sum_{x,\mu,\nu}|\mathrm{d}A(x,\mu,\nu)|^2$$
$$+ \tilde{R}_h\sum_{x,\mu,\nu}\partial_\mu A(x,\mu)\partial_\nu A(x,\nu) + \tilde{S}_h\sum_{x,\mu}(\partial_\mu A(x,\mu))^2 + \tilde{\kappa}_h\sum_{x,\mu}A(x,\mu)^4$$
$$+ \tilde{\zeta}_h\sum_{x,\mu,\nu}A(x,\mu)^2A(x,\nu)^2$$

- Non-gauge-invariant terms are spurious, and must be shown to flow to zero in the limit $h^* \to -\infty$.

RCCs

- The procedure outlined above allows us to express $V^{(h)}$ as a function of the Running Coupling Constants:

$$\mathbf{v}_h = \left\{\nu_h, e_h, \eta_h, z_h^{\psi}, z_h^{A}, m_h, R_h, S_h, \kappa_h, \zeta_h\right\}$$

- $z_h^{\psi} = \frac{Z_h^{\psi}}{Z_{h+1}^{\psi}} - 1$, $z_h^{A} = \frac{Z_h^{A}}{Z_{h+1}^{A}} - 1$.

- $\mathbf{v}_h$ is defined recursively, and thus it depends itself on $\{\mathbf{v}_k\}_{h<k\leq N}$. The recursive equations are known as Beta function equations:

$$\nu_h = 2\nu_{h+1} + B_h^{\nu}(\{\mathbf{v}_k\}_{h<k\leq N}) \qquad e_h = e_{h+1} + B_h^{e}(\{\mathbf{v}_k\}_{h<k\leq N})$$

$$z_h^{\psi} = B_h^{z^{\psi}}(\{\mathbf{v}_k\}_{h<k\leq N}) \qquad z_h^{A} = B_h^{z^{A}}(\{\mathbf{v}_k\}_{h<k\leq N})$$

$$m_h = 4m_{h+1} + B_h^{m}(\{\mathbf{v}_k\}_{h<k\leq N}) \qquad \sharp_h = \sharp_{h+1} + B_h^{\sharp}(\{\mathbf{v}_k\}_{h<k\leq N})$$

with $\sharp \in \{\eta, R, S, \kappa, \zeta\}$.

- The iteration goes on until the scale h^*. At this point the bosons are integrated out completely, and the resulting fermionic theory has an irrelevant quartic interaction. As a result, at scales $h' < h$, the RCCs are "frozen" at their values at h^*.
- The iterative procedure is non-perturbative in nature, however the kernels of $V^{(h)}$ are written as a formal power series in $\{\mathbf{v}_k\}_{h<k\leq N}$, with coefficients bounded in L^1 norm by $C^n n!$.
- Truncations of this series are sensible as long as the RCCs are small enough, uniformly in h; we will show this by studying the solution to truncations of the Beta function equations.

Flow of the Running Coupling Constants

Controlling the spurious RCCs (and electron mass)

- The flow of ν_h to zero is controlled by tuning the counterterm ν_N properly.
- WI + Explicit control of a few diagrams $\Longrightarrow$ $\sharp_{h^*} = e_{h^*} O(\varepsilon^{\lambda_\sharp})$.
- Since $\lim_{h^* \to -\infty} e_{h^*} = 0$, the non-gauge-invariant terms vanish in the IR.

The flow of physical RCCs

- Vertex WI $\Longrightarrow$ $\sqrt{Z^A_{h^*}}\, e_{h^*} = \sqrt{Z^A_N}\, e_N \left(1 + O(e^2_{h^*})\right)$.
- $z^A_{h^*} = \frac{\ln 2}{6\pi^2} e^2_{h^*} + O(e^4_{h^*})$
- $\therefore$ $Z^A_{h^*} = Z^A_N \Big(1 + \frac{\ln 2}{6\pi^2} e^2_N \big((N - h^*) + \cdots\big) + O(e^4_N)\Big)$
- $\therefore$ $e^2_{h^*} = \dfrac{e^2_N}{1 + \frac{\ln 2}{6\pi^2} e^2_N (N - h^*) + \cdots}$
- $Z^\psi_{h^*} = Z^\psi_N \left(1 + O(e^2_{h^*})\right)$ (this flow is gauge-dependent, true in Lorenz gauge)

Optimal UV cutoff

- If we fix e_0 such that $\boxed{e_0^2 = \frac{4\pi}{137} \sim 0.092}$, we get

$$e_N^2 \simeq \frac{e_0^2}{1 - \frac{\ln 2}{6\pi^2} e_0^2 N + \cdots}$$

hence to make e_N small, one needs $N \leq C e_0^{-2}$, *i.e.* $\boxed{\ell \geq \ell_0 e^{-C e_0^{-2}}}$.

Outlook

Further Steps

- Anisotropic Lattice (emergent symmetries ?)
- Non-perturbative definition of LQED: new ideas are required (Balaban-Dimock renormalization scheme ?)
- IR Lattice Electroweak Model (?)

Thank you for the attention!